Tutorial problems — MACM101 (Summer 2026), Week 3

Problem 1. Show that the following compound statements are logically equivalent:

- $(p \to r) \land (q \to r)$ and $(p \lor q) \to r$;
- $\neg p \to (q \to r)$ and $q \to (p \lor r)$.

Answer.

For the first equivalence,

$$\begin{aligned}(p \to r) \land (q \to r) &\equiv (\neg p \lor r) \land (\neg q \lor r) && \text{implication law}\\ &\equiv (\neg p \land \neg q) \lor r && \text{distributive law}\\ &\equiv \neg(p \lor q) \lor r && \text{De Morgan's law}\\ &\equiv (p \lor q) \to r. && \text{implication law}\end{aligned}$$

For the second equivalence,

$$\begin{aligned}\neg p \to (q \to r) &\equiv p \lor (\neg q \lor r) && \text{implication law}\\ &\equiv \neg q \lor (p \lor r) && \text{associative and commutative laws}\\ &\equiv q \to (p \lor r). && \text{implication law}\end{aligned}$$

Problem 2. Is $(p \lor q) \to (q \to (p \land q))$ a contradiction?

Answer.

No. We simplify:

$$\begin{aligned}(p \lor q) \to (q \to (p \land q)) &\equiv \neg(p \lor q) \lor (\neg q \lor (p \land q))\\ &\equiv (\neg p \land \neg q) \lor \neg q \lor (p \land q)\\ &\equiv \neg q \lor (p \land q)\\ &\equiv (\neg q \lor p) \land (\neg q \lor q)\\ &\equiv p \lor \neg q.\end{aligned}$$

This is false only when p is false and q is true, so it is not a contradiction.

Problem 3. Verify that

$$(p \leftrightarrow q) \land (q \leftrightarrow r) \land (r \leftrightarrow p) \Leftrightarrow (p \to q) \land (q \to r) \land (r \to p).$$

Answer.

Let

$$L = (p \leftrightarrow q) \land (q \leftrightarrow r) \land (r \leftrightarrow p)$$

and

$$R = (p \to q) \land (q \to r) \land (r \to p).$$

If L is true, then $p \to q$, $q \to r$, and $r \to p$ are true, so R is true.
Conversely, suppose R is true. Then the implications

$$p \to q, \qquad q \to r, \qquad r \to p$$

form a cycle. By going around the cycle, we also get

$$q \to p, \qquad r \to q, \qquad p \to r.$$

Therefore $p \leftrightarrow q$, $q \leftrightarrow r$, and $r \leftrightarrow p$ are all true. Hence L is true. Thus $L \leftrightarrow R$ is a tautology.

Problem 4. Negate the following statement and simplify the result:

$$p \vee q \vee (\neg p \wedge \neg q \wedge r).$$

Answer.

First,

$$\begin{aligned} p \vee q \vee (\neg p \wedge \neg q \wedge r) &\equiv (p \vee q) \vee (\neg(p \vee q) \wedge r) \\ &\equiv (p \vee q) \vee r \\ &\equiv p \vee q \vee r. \end{aligned}$$

Therefore the negation is

$$\neg(p \vee q \vee r) \equiv \neg p \wedge \neg q \wedge \neg r.$$

Problem 5. Let "Nand" be the logic connective defined by $p \uparrow q \iff \neg(p \wedge q)$. Express $\neg, \vee, \wedge$ using only Nand. (difficult problem, not for everyone)

Answer.

We have

$$\neg p \equiv p \uparrow p.$$

Therefore,

$$p \vee q \equiv \neg p \uparrow \neg q \equiv (p \uparrow p) \uparrow (q \uparrow q),$$

and

$$p \wedge q \equiv \neg(p \uparrow q) \equiv (p \uparrow q) \uparrow (p \uparrow q).$$

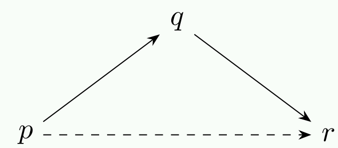
Problem 6. Verify that the Rule of Syllogism is a valid argument. (Use the corresponding tautology.)

Answer.

Some setup. Let us recall when an argument is valid. An argument is valid if it is impossible for the conclusion to be false whenever all the premises are true.
The Rule of Syllogism is the following.

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

The Rule of Syllogism says that if $p \to q$ and $q \to r$, then we get $p \to r$ for free (see the picture below).



Next, we show that the Rule of Syllogism is valid by a proof by contradiction.
Proof of validity. Suppose, for the sake of contradiction, that the Rule of Syllogism is not valid. That is, suppose the premises ($p \to q$ and $q \to r$) are true, while the conclusion ($p \to r$) is false. Since $p \to r \equiv \neg p \vee r$ is false, it follows that p must be true, and r must be false. By Modus Ponens, from $p \to q$ and p we get that q must be true. By another Modus Ponens, from $q \to r$ and q we get that r must be true. This contradicts our earlier argument that r must be false. □

Problem 7. Verify that the following is a tautology by showing that it is impossible for the conclusion to have truth value 0 while the premises have truth value 1:

$$((p \to q) \wedge (r \to s) \wedge (p \vee r)) \to (q \vee s).$$

Answer.

Suppose, toward a contradiction, that the premises are true and the conclusion $q \vee s$ is false. Then q is false and s is false.
Since $p \to q$ is true and q is false, p must be false. Since $r \to s$ is true and s is false, r must be false. But then $p \vee r$ is false, contradicting the premise $p \vee r$.
Therefore it is impossible for the premises to be true while the conclusion is false. Hence the formula is a tautology.

Problem 8. For each of the following pairs of statements use Modus Ponens or Modus Tollens to make a valid argument.

- "If Janice has trouble starting her car, then her daughter Angela will check Janice's spark plugs.
  Janice had trouble starting her car."
- "If Brady solved the first problem correctly, then the answer he obtained is 137.
  Brady's answer to the first problem is not 137."

Answer.

For the first pair, let

$$J = \text{"Janice has trouble starting her car"}, \qquad A = \text{"Angela checks Janice's spark plugs"}.$$

The premises are $J \to A$ and J. By Modus Ponens, A. Therefore Angela will check Janice's spark plugs.
For the second pair, let

$$S = \text{"Brady solved the first problem correctly"}, \qquad R = \text{"Brady's answer is 137"}.$$

The premises are $S \to R$ and $\neg R$. By Modus Tollens, $\neg S$. Therefore Brady did not solve the first problem correctly.

Problem 9. Give the reasons for each step needed to show that the following argument is valid.

Premises: $p, p \to q, s \vee r, r \to \neg q$. Conclusion: s.

Steps	Reasons
1. p	
2. $p \to q$	
3. q	
4. $r \to \neg q$	
5. $q \to \neg r$	
6. $\neg r$	
7. $s \vee r$	
8. s	

Answer.

Steps	Reasons
1. p	Premise
2. $p \to q$	Premise
3. q	Modus Ponens from 1 and 2
4. $r \to \neg q$	Premise
5. $q \to \neg r$	Contrapositive of 4
6. $\neg r$	Modus Ponens from 3 and 5
7. $s \vee r$	Premise
8. s	Disjunctive Syllogism from 6 and 7

Problem 10. Solve problems 5,6 for the previous Tutorial using rules of inference. (difficult problem, not for everyone)

Answer.

For Problem 5 from the previous tutorial, let

$$A = \text{“}A \text{ is a knight”}, \qquad B = \text{“}B \text{ is a knight”}.$$

The two statements give

$$A \leftrightarrow B, \qquad B \leftrightarrow (A \oplus B),$$

because A says that B is a knight, and B says that A and B are of opposite types.
Assume, toward a contradiction, that B is true. From $A \leftrightarrow B$, we get $B \to A$, and hence A. From $B \leftrightarrow (A \oplus B)$, we get $B \to (A \oplus B)$, and hence $A \oplus B$. But A and B cannot both be true while $A \oplus B$ is true. Therefore $\neg B$. From $A \to B$ and $\neg B$, Modus Tollens gives $\neg A$. Hence both A and B are knaves.
For Problem 6 from the previous tutorial, again let

$$A = \text{“}A \text{ is a knight”}, \qquad B = \text{“}B \text{ is a knight”}.$$

The two statements give

$$A \leftrightarrow (A \wedge B), \qquad B \leftrightarrow \neg A,$$

because A says that they are both knights, and B says that A is a knave.
Assume, toward a contradiction, that A is true. From $A \leftrightarrow (A \wedge B)$, we get $A \to (A \wedge B)$, and hence B. From $B \leftrightarrow \neg A$, we get $B \to \neg A$, and hence $\neg A$, a contradiction. Therefore $\neg A$. From $B \leftrightarrow \neg A$, we get $\neg A \to B$. By Modus Ponens, B. Hence A is a knave and B is a knight.