Tutorial solutions — MACM101 (Summer 2026), Week 4

Problem 1. Consider the universe of all polygons with three or four sides, and define the following predicates for this universe:

$A(x)$: all interior angles of x are equal;
$E(x)$: x is an equilateral triangle;
$H(x)$: all sides of x are equal;
$P(x)$: x has an interior angle that exceeds $180°$;
$Q(x)$: x is a quadrilateral;
$R(x)$: x is a rectangle;
$S(x)$: x is a square;
$T(x)$: x is a triangle.

Translate each of the following statements into an English sentence, and determine whether the statement is true or false.

1. $\forall x\ (Q(x) \oplus T(x))$;
2. $\exists x\ (T(x) \wedge P(x))$;
3. $\exists x\ (Q(x) \wedge \neg R(x))$;
4. $\forall x\ (H(x) \rightarrow E(x))$;
5. $\forall x\ (S(x) \leftrightarrow (A(x) \wedge H(x)))$.

Answer.

1. Every polygon in the universe is either a quadrilateral or a triangle, but not both. This is **true**, since the universe consists exactly of polygons with three or four sides.
2. There is a triangle with an interior angle greater than $180°$. This is **false**, since a triangle cannot have an interior angle exceeding $180°$.
3. There is a quadrilateral that is not a rectangle. This is **true**; for example, a non-rectangular parallelogram or a trapezoid works.
4. Every polygon whose sides are all equal is an equilateral triangle. This is **false**; for example, a square has all sides equal but is not an equilateral triangle.
5. A polygon is a square if and only if all its interior angles are equal and all its sides are equal. This is **false** in this universe, since an equilateral triangle also has all its angles equal and all its sides equal, but it is not a square.

Problem 2. Let $P(x,y), Q(x,y)$ denote the following predicates

$$P(x,y): \quad x^2 \geq y \qquad\qquad Q(x,y): \quad x+2 < y.$$

If the universe for each of x, y consists of all real numbers, determine the truth value for each of the following statements

1. $P(-3,8) \wedge Q(1,3)$;
2. $P(\frac{1}{2},\frac{1}{3}) \vee \neg Q(-2,-3)$;
3. $P(1,2) \leftrightarrow \neg Q(1,2)$.

Answer.

1. $P(-3, 8)$ is true because $(-3)^2 = 9 \geq 8$. But $Q(1, 3)$ is false because $1 + 2 < 3$ is $3 < 3$, which is false. Therefore the conjunction is **false**.
2. $P(\frac{1}{2}, \frac{1}{3})$ is false because $(\frac{1}{2})^2 = \frac{1}{4} \not\geq \frac{1}{3}$. Also, $Q(-2, -3)$ is false because $-2 + 2 < -3$ is $0 < -3$, which is false. Thus $\neg Q(-2, -3)$ is true, so the disjunction is **true**.
3. $P(1, 2)$ is false because $1^2 \not\geq 2$. Also, $Q(1, 2)$ is false because $1+2 < 2$ is false, so $\neg Q(1, 2)$ is true. Since false is not equivalent to true, the biconditional is **false**.

Problem 3. Let $P(x), Q(x)$, and $R(x)$ denote the following predicates

$$P(x): \quad x^2 - 8x + 15 = 0 \qquad Q(x): \quad x \text{ is odd} \qquad R(x): \quad x > 0.$$

If the universe for x consists of all integers, determine the truth value for each of the following statements. If a statement is false, give a counterexample.

1. $\forall x\ (P(x) \to Q(x))$;
2. $\exists x\ (P(x) \to Q(x))$;
3. $\exists x\ (R(x) \to P(x))$;
4. $\forall x\ ((P(x) \vee Q(x)) \to R(x))$.

Answer.

First note that

$$x^2 - 8x + 15 = (x - 3)(x - 5),$$

so $P(x)$ is true exactly when $x = 3$ or $x = 5$.

1. This is **true**. If $P(x)$ is true, then $x = 3$ or $x = 5$, and both are odd.
2. This is **true**. For example, $x = 3$ makes both $P(x)$ and $Q(x)$ true, so $P(x) \to Q(x)$ is true. We could also take $x = 4$ as our example, which makes both $P(x)$ and $Q(x)$ false, so $P(x) \to Q(x)$ true.
3. This is **true**. For example, $x = 3$ gives $R(3)$ true and $P(3)$ true, so $R(3) \to P(3)$ is true. We could also take $x = 0$, which gives $R(x)$ and $P(x)$ both false, so $R(0) \to P(0)$ is true.
4. This is **false**. Take $x = -1$. Then $Q(-1)$ is true, so $P(-1) \vee Q(-1)$ is true, but $R(-1)$ is false. Thus the implication is false.

Problem 4. Write the negation of each of the following statements as an English sentence — without symbolic notation. (Here the universe consists of all the students at the university where Professor Lenhart teaches.)

1. Every student in Professor Lenhart's C++ class is majoring in computer science or mathematics.
2. At least one student in Professor Lenhart's C++ class is a history major.

Answer.

A safe way to write the negation of such statements is to translate them first into logical forms, then negate the logical form, and then translate them back to English. Let our universe U be the students in professor Lenhart's class. We have three predicates appearing above.

Let $C(x), M(x), H(x)$ denote that student x is majoring in CS, mathematics, and history respectively. Then, the sentences become easy to negate after one writes them in logical form.

1. The first statement is $\forall x: \ C(x) \vee M(x)$. Negating it gives $\neg(\forall x: \ C(x) \vee M(x)) \equiv \exists x: \neg(C(x) \vee M(x))$. DeMorgan's law gives the full negation: $\exists x: \neg C(x) \wedge \neg M(x)$.
2. The second statement is $\exists x: \ H(x)$. Negation flips the existential to universal. $\neg(\exists x: H(x)) \equiv \forall x: \ \neg H(x)$.

Now it's a matter of reading the above logical statements out loud!

1. There is at least one student in Professor Lenhart's C++ class who is not majoring in computer science and is not majoring in mathematics.
2. No student in Professor Lenhart's C++ class is a history major.

We *can* use the suggested universe of everyone studying at the university instead, but we will need an extra predicate $P(x)$ denoting that student x is in professor Lenhart's class. With this universe,

1. The first statement is $\forall x: P(x) \to (C(x) \vee M(x))$. Negating it gives

$$\begin{aligned} \neg(\forall x: P(x) \to (C(x) \vee M(x))) &\equiv \exists x: \neg(P(x) \to (C(x) \vee M(x))) && (1)\\ &\equiv \exists x: P(x) \wedge \neg(C(x) \vee M(x)) && (2)\\ &\equiv \exists x: P(x) \wedge \neg C(x) \wedge \neg M(x) && (3)\end{aligned}$$

2. The second statement is $\exists x: P(x) \wedge H(x)$. Negating it gives

$$\begin{aligned} \neg\exists x: P(x) \wedge H(x)) &\equiv \forall x: \neg(P(x) \wedge H(x)) && (4)\\ &\equiv \forall x: \neg P(x) \vee \neg H(x) && (5)\end{aligned}$$

The English interpretations mean the same thing, but are phrased differently.

1. There is at least one student at the university where Professor Lenhart teaches who is in Professor Lenhart's C++ class, is not majoring in computer science, and is not majoring in math.

2. Everyone at the university where Professor Lenhart teaches is either not taking his C++ class, or is not a history major.

Problem 5. Negate and simplify each of the following.

1. $\forall x \ (P(x) \to Q(x))$;
2. $\exists x \ ((P(x) \vee Q(x)) \to R(x))$.

Answer.

1.

$$\begin{aligned} \neg\forall x \, (P(x) \to Q(x)) &\equiv \exists x \, \neg(P(x) \to Q(x))\\ &\equiv \exists x \, (P(x) \wedge \neg Q(x)).\end{aligned}$$

2.

$$\begin{aligned}\neg\exists x\,((P(x)\vee Q(x))\to R(x)) &\equiv \forall x\,\neg((P(x)\vee Q(x))\to R(x))\\ &\equiv \forall x\,((P(x)\vee Q(x))\wedge\neg R(x)).\end{aligned}$$

Problem 6. Determine the truth value of each statement in the case when the universe comprises all nonzero integers, and in the case when the universe consists of all nonzero real numbers.

1. $\exists x\forall y\ (xy = 1)$;
2. $\forall x\exists y\ ((2x + y = 5)\wedge(x - 3y = -8))$.

Answer.

1. This is **false** for both universes. There is no fixed nonzero integer or nonzero real number x such that $xy = 1$ for every nonzero y. When justifying this answer, note that just one counterexample does not suffice! The negation of the statement is true in both universes, and can be written and simplified like so:

$$\neg(\exists x\forall y\ (xy = 1) \equiv \forall x\exists y\ xy \neq 1 \tag{6}$$

Let's prove this for the universe of all nonzero integers:

Proof. Suppose **incomplete**

> Here is the subproof or variable context. Let $x = 2$. Then $2 + y = z$.

□

For example, $xy = 1$ cannot hold for both $y = 1$ and $y = 2$ using the same x.

2. This is **false** for both universes. Solving the two equations gives

$$y = 5 - 2x \qquad \text{and} \qquad y = \frac{x+8}{3}.$$

At this point it is clear that the statement cannot be true if the universe is the set of integers, because $y = \frac{x+8}{3}$ is not an integer for all $x \in \mathbb{Z}$. On the other hand, for the reals, the values of y must coincide, so we must have

$$5 - 2x = \frac{x+8}{3},$$

so $15 - 6x = x + 8$, hence $x = 1$. Therefore the system can only work when $x = 1$ (and *not all* x).

Problem 7. Let the universe for the variables in the following statements consists of all real numbers. In each case negate and simplify the given statement.

1. $\forall x\forall y\ ((x < y) \to \exists z\ (x < z < y))$;
2. $(\forall x\forall y\ ((x > 0)\wedge(y > 0))) \to (\exists z\ (xz > y))$.

Answer.

1. $$\neg\forall x\forall y\,((x<y)\to\exists z\,(x<z<y))$$

   becomes

   $$\exists x\exists y\,\big((x<y)\wedge\forall z\,\neg(x<z<y)\big).$$

   Equivalently,

   $$\exists x\exists y\,\big((x<y)\wedge\forall z\,((z\le x)\vee(z\ge y))\big).$$

2. As written, the formula has free occurrences of x and y in the consequent. Assuming the intended statement is

   $$\forall x\forall y\,\big(((x>0)\wedge(y>0))\to\exists z\,(xz>y)\big),$$

   its negation simplifies to

   $$\exists x\exists y\,\big((x>0)\wedge(y>0)\wedge\forall z\,(xz\le y)\big).$$