Tutorial solutions — MACM101 (Summer 2026), Week 5

Problem 1. Negate and simplify each of the following.

1. $\forall x\ (P(x) \to Q(x))$;
2. $\exists x\ ((P(x) \vee Q(x)) \to R(x))$.

Answer.

1. $$\neg\forall x\ (P(x) \to Q(x)) \equiv \exists x\ \neg(P(x) \to Q(x)) \equiv \exists x\ (P(x) \wedge \neg Q(x)).$$
2. $$\neg\exists x\ ((P(x)\vee Q(x)) \to R(x)) \equiv \forall x\ \neg((P(x)\vee Q(x)) \to R(x)) \equiv \forall x\ ((P(x)\vee Q(x))\wedge\neg R(x)).$$

Problem 2. Let the universe for the variables in the following statements consist of all real numbers. In each case negate and simplify the given statement.

1. $\forall x\forall y\ ((x < y) \to \exists z\ (x < z < y))$;
2. $\forall x\forall y\ \ (((x > 0) \wedge (y > 0)) \to (\exists z\ (xz > y)))$.

Answer.

1. $$\neg\forall x\forall y\ ((x < y) \to \exists z\ (x < z < y)) \equiv \exists x\exists y\ \big((x < y) \wedge \forall z\ \neg(x < z < y)\big).$$
   Equivalently,
   $$\exists x\exists y\ \big((x < y) \wedge \forall z\ ((z \le x) \vee (z \ge y))\big).$$
2. $$\begin{aligned} &\neg\forall x\forall y\ \ (((x > 0) \wedge (y > 0)) \to \exists z\ (xz > y)) \\ &\equiv \exists x\exists y\ \neg\,(((x > 0) \wedge (y > 0)) \to \exists z\ (xz > y)) \\ &\equiv \exists x\exists y\ \ ((x > 0) \wedge (y > 0) \wedge \forall z\ (xz \le y))\,. \end{aligned}$$

Problem 3. Determine which of the following arguments are valid and which are invalid. Provide an explanation for each answer. (Let the universe consist of all people presently residing in Canada.)

1. All mail carriers carry a can of mace.
   Mrs. Bacon is a mail carrier.
   Therefore Mrs. Bacon carries a can of mace.

2. All law-abiding citizens pay their taxes.
   Mr. Pelosi pays his taxes.
   Therefore Mr. Pelosi is a law-abiding citizen.

3. All people who are concerned about the environment recycle their plastic containers.
   Margarite is not concerned about the environment.
   Therefore Margarite does not recycle her plastic containers.

Answer.

1. Valid. If every mail carrier carries a can of mace and Mrs. Bacon is a mail carrier, then by universal instantiation and modus ponens, Mrs. Bacon carries a can of mace.
2. Invalid. The premises say that being law-abiding implies paying taxes, but they do not say that everyone who pays taxes is law-abiding. This is the converse error.
3. Invalid. The premises say that being concerned about the environment implies recycling plastic containers, but they do not say that people who are not concerned fail to recycle. This is the inverse error.

Problem 4. For a prescribed universe and any propositions $P(x), Q(x)$, prove that

$$\forall x\ (P(x) \wedge Q(x)) \quad \Longleftrightarrow \quad (\forall x\ P(x)) \wedge (\forall x\ Q(x))\,.$$

Answer.

We prove both directions.
First suppose that

$$\forall x\ (P(x) \wedge Q(x)).$$

Let a be an arbitrary element of the universe. Then $P(a) \wedge Q(a)$, so $P(a)$ and $Q(a)$. Since a was arbitrary, $\forall x\ P(x)$ and $\forall x\ Q(x)$. Therefore

$$(\forall x\ P(x)) \wedge (\forall x\ Q(x)).$$

Conversely, suppose that

$$(\forall x\ P(x)) \wedge (\forall x\ Q(x)).$$

Then $\forall x\ P(x)$ and $\forall x\ Q(x)$. Let a be an arbitrary element of the universe. Then $P(a)$ and $Q(a)$, so $P(a) \wedge Q(a)$. Since a was arbitrary,

$$\forall x\ (P(x) \wedge Q(x)).$$

Thus the two statements are logically equivalent.

Problem 5. Provide the reasons for the steps verifying the following argument. (Here a denotes a specific but arbitrary chosen element from the given universe.) (For Step 9 refer to Table 1 (Section 1.6) in the textbook.)

Premises: $\forall x\ (P(x) \to (Q(x) \wedge R(x)))$, $\forall x\ (P(x) \wedge S(x))$. Conclusion: $\forall x\ (R(x) \wedge S(x))$.

Steps	Reasons
1. $\forall x\ (P(x) \to (Q(x) \land R(x)))$	
2. $\forall x\ (P(x) \land S(x))$	
3. $P(a) \to (Q(a) \land R(a))$	
4. $P(a) \land S(a)$	
5. $P(a)$	
6. $Q(a) \land R(a)$	
7. $R(a)$	
8. $S(a)$	
9. $R(a) \land S(a)$	
10. $\forall x\ (R(x) \land S(x))$.	

Answer.

Steps	Reasons
1. $\forall x\ (P(x) \to (Q(x) \land R(x)))$	Premise
2. $\forall x\ (P(x) \land S(x))$	Premise
3. $P(a) \to (Q(a) \land R(a))$	Universal instantiation from 1
4. $P(a) \land S(a)$	Universal instantiation from 2
5. $P(a)$	Simplification from 4
6. $Q(a) \land R(a)$	Modus ponens from 3 and 5
7. $R(a)$	Simplification from 6
8. $S(a)$	Simplification from 4
9. $R(a) \land S(a)$	Conjunction from 7 and 8
10. $\forall x\ (R(x) \land S(x))$	Universal generalization from 9

Problem 6. Give a direct proof that, for all integers k and ℓ, if k, ℓ are both even, then $k + \ell$ is even.

Answer.

Suppose k and ℓ are both even. Then there exist integers m and n such that

$$k = 2m \qquad \text{and} \qquad \ell = 2n.$$

Therefore

$$k + \ell = 2m + 2n = 2(m + n).$$

Multiplying any integer (in this case $m + n$) by 2 gives an even number. Therefore, $k + \ell$ is even.

Problem 7. Give a proof by contraposition that, for all integers k and ℓ, if $k\ell$ is odd, then k, ℓ are both odd.

Answer.

We prove the contrapositive. The contrapositive statement is: *if k and ℓ are not both odd, then $k\ell$ is not odd*. (Or equivalently, *if at least one of k or ℓ is even, then $k\ell$ is even*).
Suppose k and ℓ are not both odd. Then at least one of them is even. If k is even, then $k = 2m$ for some integer m, and therefore

$$k\ell = (2m)\ell = 2(m\ell),$$

so $k\ell$ is even. If ℓ is even, then $\ell = 2n$ for some integer n, and therefore

$$k\ell = k(2n) = 2(kn),$$

so $k\ell$ is even. In either case, $k\ell$ is not odd.
Thus the contrapositive is true, so the original statement is true.
Comment: The role of k and ℓ in this proof is symmetric: if you rename k to ℓ and ℓ to k, the statement would still mean the same. Therefore, only one of the cases is enough to check. So one could say something like "The case where ℓ is even can be handled similarly as k" and avoid writing the second half of this proof.