Problems to Week 6 Tutorial — MACM 101 (Spring 2025)

Problem 1. Let $A = \{1, \{1\}, \{2\}\}$. Which of the following statements are true?

$a)$	$1 \in A$	$b)$	$\{1\} \in A$
$c)$	$\{1\} \subseteq A$	$d)$	$\{\{1\}\} \subseteq A$
$e)$	$\{2\} \in A$	$f)$	$\{2\} \subseteq A$
$g)$	$\{\{2\}\} \subseteq A$		

Answer.

The elements of A are 1, $\{1\}$, and $\{2\}$. So

	a	b	c	d	e	f	g
True?	T	T	T	T	T	F	T

The only false one is f, because $\{2\} \subseteq A$ would require $2 \in A$, but 2 is not an element of A. (Compare f with g, and try spotting the difference.)

Problem 2. For $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ let

$$A = \{1, 2, 3, 4, 5\}, \qquad B = \{1, 2, 4, 8\}, \qquad C = \{1, 2, 3, 5, 7\}, \qquad D = \{2, 4, 6, 8\}.$$

Determine each of the following:

$a)$	$(A \cup B) \cap C$	$b)$	$A \cup (B \cap C)$
$c)$	$\overline{C} \cup \overline{D}$	$d)$	$\overline{C \cap D}$
$e)$	$(A \cup B) - C$	$f)$	$A \cup (B - C)$
$g)$	$(B - C) - D$	$h)$	$B - (C - D)$
$i)$	$(A \cup B) - (C \cap D)$		

Draw Venn diagrams for each of the expressions.

Answer.

We compute the parts one at a time.

(a) We want $(A \cup B) \cap C$. First,

$$A \cup B = \{1, 2, 3, 4, 5\} \cup \{1, 2, 4, 8\} = \{1, 2, 3, 4, 5, 8\}.$$

Therefore

$$(A \cup B) \cap C = \{1, 2, 3, 4, 5, 8\} \cap \{1, 2, 3, 5, 7\} = \{1, 2, 3, 5\}.$$

(b) We want $A \cup (B \cap C)$. First,

$$B \cap C = \{1, 2, 4, 8\} \cap \{1, 2, 3, 5, 7\} = \{1, 2\}.$$

Therefore

$$A \cup (B \cap C) = \{1, 2, 3, 4, 5\} \cup \{1, 2\} = \{1, 2, 3, 4, 5\}.$$

(c) We want $\overline{C} \cup \overline{D}$. We have

$$\overline{C} = \{4, 6, 8, 9, 10\}, \qquad \overline{D} = \{1, 3, 5, 7, 9, 10\}.$$

Thus

$$\overline{C} \cup \overline{D} = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

(d) We want $\overline{C \cap D}$. First,

$$C \cap D = \{1, 2, 3, 5, 7\} \cap \{2, 4, 6, 8\} = \{2\}.$$

Therefore

$$\overline{C \cap D} = U - \{2\} = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

(e) We want $(A \cup B) - C$. From part (a),

$$A \cup B = \{1, 2, 3, 4, 5, 8\}.$$

So

$$(A \cup B) - C = \{1, 2, 3, 4, 5, 8\} - \{1, 2, 3, 5, 7\} = \{4, 8\}.$$

(f) We want $A \cup (B - C)$. First,

$$B - C = \{1, 2, 4, 8\} - \{1, 2, 3, 5, 7\} = \{4, 8\}.$$

Therefore

$$A \cup (B - C) = \{1, 2, 3, 4, 5\} \cup \{4, 8\} = \{1, 2, 3, 4, 5, 8\}.$$

(g) We want $(B - C) - D$. From part (f),

$$B - C = \{4, 8\}.$$

Thus

$$(B - C) - D = \{4, 8\} - \{2, 4, 6, 8\} = \emptyset.$$

(h) We want $B - (C - D)$. First,

$$C - D = \{1, 2, 3, 5, 7\} - \{2, 4, 6, 8\} = \{1, 3, 5, 7\}.$$

Therefore

$$B - (C - D) = \{1, 2, 4, 8\} - \{1, 3, 5, 7\} = \{2, 4, 8\}.$$

(i) We want $(A \cup B) - (C \cap D)$. From above,

$$A \cup B = \{1, 2, 3, 4, 5, 8\}, \qquad C \cap D = \{2\}.$$

So

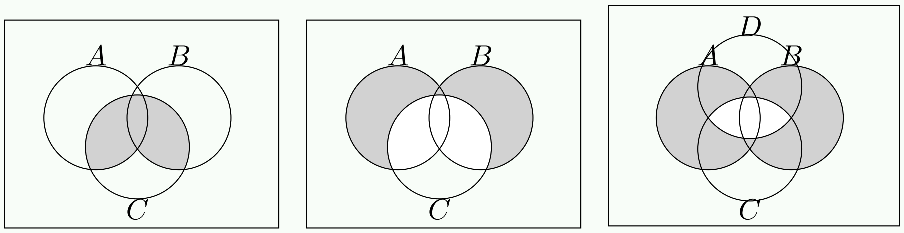
$$(A \cup B) - (C \cap D) = \{1, 2, 3, 4, 5, 8\} - \{2\} = \{1, 3, 4, 5, 8\}.$$

Hence the final answers are

$a)$	$\{1, 2, 3, 5\}$	$b)$	$\{1, 2, 3, 4, 5\}$
$c)$	$\{1, 3, 4, 5, 6, 7, 8, 9, 10\}$	$d)$	$\{1, 3, 4, 5, 6, 7, 8, 9, 10\}$
$e)$	$\{4, 8\}$	$f)$	$\{1, 2, 3, 4, 5, 8\}$
$g)$	$\emptyset$	$h)$	$\{2, 4, 8\}$
$i)$	$\{1, 3, 4, 5, 8\}$.		

Here are three Venn diagrams. In each diagram, the shaded region (in gray) is the expression we are computing.

(a) $(A \cup B) \cap C$ (e) $(A \cup B) - C$ (i) $(A \cup B) - (C \cap D)$



The remaining Venn diagrams are drawn similarly: shade the region described by the expression, and then keep exactly the elements of U which lie in that shaded region.

Problem 3. Demonstrate each of the following using Venn diagrams. Assume a universe U.

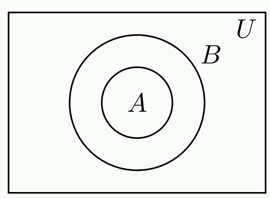
(a) If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$ and $A \cup C \subseteq B \cup D$.

(b) $A \subseteq B$ if and only if $A \cap \overline{B} = \emptyset$.

(c) $A \subseteq B$ if and only if $\overline{A} \cup B = U$.

Answer.

For (a), draw A inside B and C inside D. Then the part common to both A and C is automatically inside the part common to both B and D. Similarly, anything in $A \cup C$ is either in A or in C, hence it is either in B or in D, so it lies in $B \cup D$.
For (b), the picture is the following kind of picture:



If A is inside B, then no point of A is outside B, so $A \cap \overline{B} = \emptyset$. Conversely, if $A \cap \overline{B} = \emptyset$, then there is no point of A outside B, so $A \subseteq B$.
For (c), if $A \subseteq B$, then every point is either outside A or inside B. Hence $\overline{A} \cup B = U$. Conversely, if $\overline{A} \cup B = U$, then there is no point which is in A and outside B, so $A \subseteq B$.

Problem 4. Prove or disprove each of the following:

(a) For sets $A, B, C \subseteq U$, if $A \cup C = B \cup C$, then $A = B$.

(b) For sets $A, B, C \subseteq U$, if $A \cup C = B \cup C$ and $A \cap C = B \cap C$, then $A = B$.

Answer.

(a) This is false. We give a counterexample. Take $U = \{1\}$, $A = \{1\}$, $B = \emptyset$, and $C = \{1\}$. Then

$$A \cup C = \{1\} = B \cup C,$$

but $A \neq B$. (There are many counterexamples of this kind. Simply take any non-empty set C, and define A and B to be unequal subsets of C.)

(b) This is true. We need to show that $A = B$ follows from the mentioned assumptions. Generally to show the equality of sets $A = B$, we need to prove that $A \subseteq B$ and $B \subseteq A$. Let $x \in A$. If $x \in C$, then $x \in A \cap C = B \cap C$, so $x \in B$. If $x \notin C$, then $x \in A \cup C = B \cup C$, and since $x \notin C$, we must have $x \in B$. Thus $A \subseteq B$. The same argument with A and B switched gives $B \subseteq A$. Therefore $A = B$.

Problem 5. Prove that $A - B = A \cap \overline{B}$.

Answer.

Let x be arbitrary. Then

$$x \in A - B \iff x \in A \text{ and } x \notin B \iff x \in A \text{ and } x \in \overline{B} \iff x \in A \cap \overline{B}.$$

Therefore $A - B = A \cap \overline{B}$.

Comment: Note that we proved two statements at the same time:

1. $A - B \subseteq A \cap \overline{B}$,
2. $A \cap \overline{B} \subseteq A - B$.

If you follow the arrows from left to right ($\Longrightarrow$), it is a proof of the Item (1), while following the arrows from right to left ($\Longleftarrow$) is a proof of Item (2). This is why "if and only if" statements are powerful.

Problem 6. Prove that $A \Delta B = (\overline{A} \cap B) \cup (A \cap \overline{B})$.

Answer.

By definition, $A \Delta B$ is the set of elements which are in exactly one of A and B. So an element of $A \Delta B$ is either in B but not in A, or in A but not in B. In symbols,

$$A \Delta B = (B - A) \cup (A - B).$$

Using $X - Y = X \cap \overline{Y}$, we get

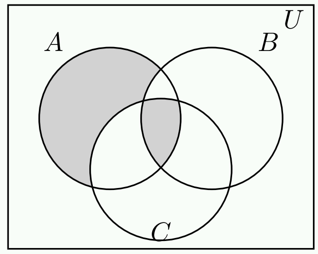
$$A \Delta B = (B \cap \overline{A}) \cup (A \cap \overline{B}) = (\overline{A} \cap B) \cup (A \cap \overline{B}).$$

Problem 7. Investigate the truth or falsity of the following using three methods: Venn diagrams, laws of set theory, and proving that the left side is a subset of the right side and vice versa.

$$A - (B \cup C) = (A - B) \cap (A - C).$$

Answer.

The statement is true.
Venn diagram. Both sides shade the part of A which is outside both B and C:



Laws of set theory. Using $X - Y = X \cap \overline{Y}$ and De Morgan's law,

$$A - (B \cup C) = A \cap \overline{(B \cup C)} = A \cap (\overline{B} \cap \overline{C}) = (A \cap \overline{B}) \cap (A \cap \overline{C}) = (A - B) \cap (A - C).$$

Subset proof. First suppose $x \in A - (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Hence $x \notin B$ and $x \notin C$. So $x \in A - B$ and $x \in A - C$, which means $x \in (A - B) \cap (A - C)$.
Conversely, suppose $x \in (A - B) \cap (A - C)$. Then $x \in A - B$ and $x \in A - C$. Hence $x \in A$, $x \notin B$, and $x \notin C$. Therefore $x \notin B \cup C$, so $x \in A - (B \cup C)$. Thus the two sets are equal.