Problems to Week 7 Tutorial — MACM 101 (Spring 2025)

Problem 1. Let A, B, C, D be nonempty sets. Prove that

$$A \times B \subseteq C \times D$$

if and only if $A \subseteq C$ and $B \subseteq D$.

Answer.

Suppose $A \times B \subseteq C \times D$. Since B is nonempty, choose $b \in B$. If $a \in A$, then $(a, b) \in A \times B$, so $(a, b) \in C \times D$. Hence $a \in C$, so $A \subseteq C$. Similarly, since A is nonempty, choosing $a \in A$ shows that every $b \in B$ belongs to D. Thus $B \subseteq D$.
Conversely, if $A \subseteq C$ and $B \subseteq D$, then for any $(a, b) \in A \times B$, we have $a \in C$ and $b \in D$. Hence $(a, b) \in C \times D$, so $A \times B \subseteq C \times D$.

Problem 2. Prove that

$$A \times (B - C) \subseteq (A \times B) - (A \times C).$$

Does equality hold?

Answer.

Let $(a, b) \in A \times (B - C)$. Then $a \in A$, $b \in B$, and $b \notin C$. Hence $(a, b) \in A \times B$ and $(a, b) \notin A \times C$. Therefore

$$(a, b) \in (A \times B) - (A \times C).$$

So

$$A \times (B - C) \subseteq (A \times B) - (A \times C).$$

In fact, equality holds. If $(a, b) \in (A \times B) - (A \times C)$, then $a \in A$, $b \in B$, and $(a, b) \notin A \times C$. Since $a \in A$, this means $b \notin C$. Thus $b \in B - C$, so $(a, b) \in A \times (B - C)$.

Problem 3. Determine which of the following relations R on the set A are reflexive, symmetric, transitive, and anti-symmetric.

1. $A = \{1, 2, 3\}$ and

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$

   Draw the graph and the matrix of this relation.

2. A is the set of all students at SFU, and $(x, y) \in R$ means that the height of x differs from the height of y by no more than one inch.

3. $A = \mathbb{R} \times \mathbb{R}$, and

$$((x_1, x_2), (y_1, y_2)) \in R$$

   if and only if $x_1 = y_1$ and $x_2 \leq y_2$.

4. Prove that the following relation is an order on

$$\{1, 2, 3\} \times \{1, 2, 3\}$$

   and draw its diagram:

$$((x_1, x_2), (y_1, y_2)) \in R$$

   if and only if $x_1 < y_1$, or $x_1 = y_1$ and $x_2 \leq y_2$.

5. Prove that the following relation on the set of all nonempty subsets of $\{a, b, c, d\}$ is an order, draw its diagram, and find all maximal, minimal, least, and greatest elements:

$$(x, y) \in R$$

if and only if $x \subseteq y$.

Answer.

1. The relation is not reflexive because $(3, 3) \notin R$.

   It is symmetric because whenever $(x, y) \in R$, the pair (y, x) is also in R.

   It is transitive. The only elements related to anything are 1 and 2, and on $\{1, 2\}$ the relation contains all possible ordered pairs.

   It is not anti-symmetric because $(1, 2) \in R$ and $(2, 1) \in R$, but $1 \neq 2$.

   The graph has loops at 1 and 2, arrows $1 \to 2$ and $2 \to 1$, and the vertex 3 is isolated.

   Using the order $1, 2, 3$, the matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. The relation is reflexive, because every student has height difference 0 from themselves.

   It is symmetric, because if the height of x differs from the height of y by at most one inch, then the same is true in the other direction.

   It is not transitive. For example, if three students have heights 60, 61, and 62 inches, then the first is related to the second and the second is related to the third, but the first is not related to the third.

   It is not anti-symmetric, since two different students can have the same height or heights differing by at most one inch.

3. The relation is reflexive because $x_1 = x_1$ and $x_2 \leq x_2$.

   It is not symmetric. For example, $((0, 1), (0, 2)) \in R$, but $((0, 2), (0, 1)) \notin R$.

   It is transitive. If

$$(x_1, x_2)R(y_1, y_2) \quad \text{and} \quad (y_1, y_2)R(z_1, z_2),$$

   then $x_1 = y_1 = z_1$ and $x_2 \leq y_2 \leq z_2$. Hence $x_2 \leq z_2$, so $(x_1, x_2)R(z_1, z_2)$.

   It is anti-symmetric. If both $(x_1, x_2)R(y_1, y_2)$ and $(y_1, y_2)R(x_1, x_2)$ hold, then $x_1 = y_1$ and $x_2 = y_2$. Hence the two ordered pairs are equal.

4. This is the lexicographic order. It is reflexive because

$$x_1 = x_1 \quad \text{and} \quad x_2 \leq x_2.$$

   It is anti-symmetric: if xRy and yRx, then neither first coordinate can be strictly smaller than the other. Thus the first coordinates are equal, and then anti-symmetry of $\leq$ gives equality of the second coordinates.

It is transitive by checking the cases in the definition. If $x_1 < y_1$ and $y_1 \leq z_1$, then $x_1 < z_1$. If all first coordinates are equal, then transitivity follows from $x_2 \leq y_2 \leq z_2$.

Thus it is an order.

Its Hasse diagram is the chain

$$(1,1) < (1,2) < (1,3) < (2,1) < (2,2) < (2,3) < (3,1) < (3,2) < (3,3).$$

5. The relation is subset inclusion. It is reflexive because $x \subseteq x$.

   It is anti-symmetric because if $x \subseteq y$ and $y \subseteq x$, then $x = y$.

   It is transitive because if $x \subseteq y$ and $y \subseteq z$, then $x \subseteq z$.

   Therefore it is an order.

   The Hasse diagram is the Boolean lattice on $\{a,b,c,d\}$ with the empty set removed. The levels are:
   $$\{a,b,c,d\},$$
   then
   $$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\},$$
   then
   $$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\},$$
   then
   $$\{a\}, \{b\}, \{c\}, \{d\}.$$

   Edges connect sets that differ by one element.

   The minimal elements are
   $$\{a\}, \{b\}, \{c\}, \{d\}.$$
   The only maximal element is
   $$\{a,b,c,d\}.$$
   There is no least element, because the empty set is not included. The greatest element is
   $$\{a,b,c,d\}.$$

Problem 4. Check that the following relations R on the set A are equivalence relations, find their equivalence classes, the number of equivalence classes, and determine which equivalence class the element z belongs to.

1. Let A be the set of all possible strings of 3 or 4 letters in the alphabet $\{A,B,C,D\}$, let $z = BCAD$, and let $(x,y) \in R$ if and only if x and y have the same first letter and the same third letter.

2. Let A be the power set of $\{1,2,3,4,5\}$, let $z = \{1,2,3\}$, and let $(x,y) \in R$ if and only if
   $$x \cap \{1,3,5\} = y \cap \{1,3,5\}.$$

Answer.

1. The relation is reflexive, symmetric, and transitive because "having the same first letter and the same third letter" has all three properties. Hence R is an equivalence relation.

   Let $\Sigma = \{A, B, C, D\}$. The equivalence classes are determined by a choice of first letter $u \in \Sigma$ and third letter $v \in \Sigma$. Thus the classes are

   $$C_{u,v} = \{x \in A : \text{the first letter of } x \text{ is } u \text{ and the third letter of } x \text{ is } v\}.$$

   There are $4 \cdot 4 = 16$ equivalence classes.

   Since $z = BCAD$ has first letter B and third letter A, we have

   $$[z] = C_{B,A}.$$

   Equivalently,

   $$[z] = \{B\ell A : \ell \in \Sigma\} \cup \{B\ell Am : \ell, m \in \Sigma\}.$$

2. The relation is reflexive, symmetric, and transitive because equality of intersections with $\{1, 3, 5\}$ has all three properties. Hence R is an equivalence relation.

   The equivalence classes are determined by subsets of $\{1, 3, 5\}$. For each $T \subseteq \{1, 3, 5\}$, the corresponding class is

   $$C_T = \{x \subseteq \{1, 2, 3, 4, 5\} : x \cap \{1, 3, 5\} = T\}.$$

   There are

   $$2^3 = 8$$

   equivalence classes.

   Since

   $$z \cap \{1, 3, 5\} = \{1, 3\},$$

   we have

   $$[z] = C_{\{1,3\}}.$$

   Explicitly,

   $$[z] = \{\{1, 3\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}.$$