Problems to Week 8 Tutorial — MACM 101 (Spring 2025)

Problem 1. Determine whether or not the following relations are functions. If a relation is a function, find its range.

(a) $\{(x, y) \mid x, y \in \mathbb{Z},\ y = x^2 + 7\}$, a relation from $\mathbb{Z}$ to $\mathbb{Z}$.

(b) $\{(x, y) \mid x, y \in \mathbb{R},\ y^2 = x\}$, a relation from $\mathbb{R}$ to $\mathbb{R}$.

Answer.

(a) This is a function. For each $x \in \mathbb{Z}$, there is exactly one value $y = x^2 + 7 \in \mathbb{Z}$.
Its range is

$$\{x^2 + 7 : x \in \mathbb{Z}\} = \{0 + 7, 1 + 7, 4 + 7, 9 + 7, 16 + 7, 25 + 7, \dots\}.$$

(b) This is not a function. For example, when $x = 1$, we have $y^2 = 1$, so both $y = 1$ and $y = -1$ are possible outputs. Thus one input can have two outputs.
Also, when $x < 0$, there is no real number y such that $y^2 = x$.

Problem 2. For each of the following functions, determine whether it is one-to-one and determine its range.

(a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$.

(b) $f : \mathbb{Q} \to \mathbb{Q}$, $f(x) = 2x + 1$.

(c) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^3 - x$.

Answer.

(a) The function is one-to-one. Indeed[a], if $2x + 1 = 2x' + 1$, then $2x = 2x'$ and finally $x = x'$.
The range is the set of odd integers:

$$\{2x + 1 : x \in \mathbb{Z}\}.$$

(b) The function is one-to-one. Indeed, if $2x + 1 = 2y + 1$, then $x = y$.
The range is all of $\mathbb{Q}$. Given any $q \in \mathbb{Q}$, choose

$$x = \frac{q - 1}{2} \in \mathbb{Q}.$$

Then $f(x) = q$.

(c) The function is not one-to-one, since[b]

$$f(-1) = (-1)^3 - (-1) = 0, \qquad f(0) = 0^3 - 0 = 0,$$

but $-1 \neq 0$.
The range is

$$\{x^3 - x : x \in \mathbb{Z}\}.$$

Equivalently, this is the set of integers of the form $x(x-1)(x+1)$, which are precisely integers obtained by multiplying three consecutive integers.

[a] Recall that usually when we check if a function f is one-to-one we need to establish that whenever the value of f on two inputs x, x' is the same, then it must follow that the inputs were the same $x = x'$.

[b] Generally, to show that function f is **not** one-to-one, one can provide two distinct inputs $x \neq x'$ for which $f(x) = f(x')$.

Problem 3. Let $f\colon A \to B$ where $A = X \cup Y$ with $X \cap Y = \emptyset$. If $f|_X$ and $f|_Y$ are one-to-one, does it follow that f is one-to-one?

Answer.

No. The restrictions being one-to-one only prevents collisions inside X and inside Y. It does not prevent an element of X and an element of Y from having the same image.
For example, let

$$X = \{1\}, \qquad Y = \{2\}, \qquad A = \{1, 2\}, \qquad B = \{0\},$$

and define $f(1) = 0$ and $f(2) = 0$. Then $f|_X$ and $f|_Y$ are both one-to-one, but f is not one-to-one, since $1 \neq 2$ and $f(1) = f(2)$.

Problem 4. For each of the following functions $f\colon \mathbb{Z} \to \mathbb{Z}$, determine whether the function is onto. If the function is not onto, determine the range of f.

(a) $f(x) = 2x - 3$.

(b) $f(x) = x^2 + x$.

Answer.

(a) The function is not onto. Since $2x - 3$ is always odd, no even integer is in the range.
The range is the set of odd integers:

$$\{2x - 3 : x \in \mathbb{Z}\} = \{\text{odd integers}\}.$$

(b) The function is not onto. Since

$$x^2 + x = x(x + 1),$$

the value is always even (because either x or $x + 1$ is always even). Thus no odd integer is in the range.
The range is

$$\{x^2 + x : x \in \mathbb{Z}\},$$

which is the set of integers obtained by multiplying two adjacent integers.

Problem 5. Let $f, g\colon \mathbb{R} \to \mathbb{R}$, where $g(x) = 1 - x + x^2$ and $f(x) = ax + b$. If $(g \circ f)(x) = 9x^2 - 9x + 3$, determine a, b.

Answer.

We have

$$(g \circ f)(x) = g(ax + b) = 1 - (ax + b) + (ax + b)^2.$$

Expanding gives

$$(g \circ f)(x) = a^2x^2 + a(2b - 1)x + (b^2 - b + 1).$$

Comparing coefficients with $9x^2 - 9x + 3$, we get

$$a^2 = 9, \qquad a(2b - 1) = -9, \qquad b^2 - b + 1 = 3.$$

From $b^2 - b + 1 = 3$, we get

$$b^2 - b - 2 = 0,$$

so $b = 2$ or $b = -1$.
If $b = 2$, then $a(3) = -9$, so $a = -3$. If $b = -1$, then $a(-3) = -9$, so $a = 3$.
Therefore,

$$(a, b) = (-3, 2) \quad \text{or} \quad (a, b) = (3, -1).$$

Problem 6. Let $g\colon \mathbb{N} \to \mathbb{N}$ be defined by $g(n) = 2n$. If $A = \{1, 2, 3, 4\}$ and $f\colon A \to \mathbb{N}$ is given by $f = \{(1, 2), (2, 3), (3, 5), (4, 7)\}$, find $g \circ f$.

Answer.

We compute $g(f(x))$ for each $x \in A$:

$$f(1) = 2, \qquad f(2) = 3, \qquad f(3) = 5, \qquad f(4) = 7.$$

Since $g(n) = 2n$, we get

$$g(f(1)) = 4,$$
$$g(f(2)) = 6,$$
$$g(f(3)) = 10,$$
$$g(f(4)) = 14.$$

Therefore,

$$g \circ f = \{(1, 4), (2, 6), (3, 10), (4, 14)\}.$$

Problem 7. If $f \circ g$ is onto, does it follow that g is onto?

Answer.

No. We provide an counterexample.
Let $C = \{1\}, B = \{a, b\}$ and let $A = \{x, y, z\}$.
Let $f : B \to C$ simply output 1 every time. Let $g : A \to B$ simply output a every time.
Therefore, the value $b \in B$ is missing from the image of g, however $f \circ g : A \to C$ is onto (because it always outputs 1 for every input, and 1 is the only element of C).
Comment: Try to show that if $f \circ g$ is onto then f must be onto as well.

Problem 8. For each of the following functions $f\colon \mathbb{R} \to \mathbb{R}$, determine whether f is invertible, and, if so, determine f^{-1}.

(a) $f = \{(x, y) \mid 2x + 3y = 7\}$.

(b) $f = \{(x, y) \mid y = x^4 + x\}$.

Answer.

Recall that a function is invertible if and only if it is a bijection (one-to-one *and* onto).
(a) Solving for y, we get

$$3y = 7 - 2x,$$

so

$$f(x) = \frac{7 - 2x}{3}.$$

This function is invertible because it is a linear function with nonzero slope.[a]

To find the inverse, set

$$y = \frac{7 - 2x}{3}.$$

Then

$$3y = 7 - 2x,$$

so

$$2x = 7 - 3y,$$

and hence

$$x = \frac{7 - 3y}{2}.$$

Therefore,

$$f^{-1}(x) = \frac{7 - 3x}{2}.$$

(b) The function is not invertible, because it is not one-to-one. Indeed,

$$f(-1) = (-1)^4 + (-1) = 0,$$

and

$$f(0) = 0^4 + 0 = 0.$$

Since $-1 \neq 0$ but $f(-1) = f(0)$, the function is not one-to-one. Therefore it is not invertible.

[a] Strictly speaking, you need to check $f(x)$ is one-to-one and onto.