Tip (Structure of an induction proof).

This tutorial is about induction. Usually we want to prove a statement $P(n)$ for natural numbers n.
Every proof by induction generally has three steps.

1. Base case(s): Start by proving the statement for the first relevant value(s) of n.
2. Induction hypothesis: Assume that the statement is true for some arbitrary natural number k. That is, assume $P(k)$ is true. In *strong* induction we assume $P(1), P(2), \ldots, P(k)$ are true.
3. Inductive step: Using the induction hypothesis, prove that $P(k+1)$ is true. In other words, prove the implication
$$P(k) \longrightarrow P(k+1).$$

The important point is that in the inductive step, we are not proving $P(k)$ from scratch, and we are not proving $P(k+1)$ in isolation. We are proving that if $P(k)$ is true, then $P(k+1)$ must also be true.

Problem 1. Prove each of the following for all $n \geq 1$ using the principle of mathematical induction.

1. $$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$
2. $$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$
3. $$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

Answer.

1. We prove the statement by induction on n.

   For $n = 1$, the left-hand side is $1^2 = 1$, and the right-hand side is
$$\frac{1(2 \cdot 1 - 1)(2 \cdot 1 + 1)}{3} = \frac{1 \cdot 1 \cdot 3}{3} = 1.$$

   So the statement is true for $n = 1$.

   Now assume that the statement is true for some $n \geq 1$, that is,
$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

We prove it for $n+1$. We have

$$\begin{aligned}
1^2 + 3^2 + \cdots + (2n-1)^2 + (2n+1)^2 &= \frac{n(2n-1)(2n+1)}{3} + (2n+1)^2 \\
&= (2n+1)\left(\frac{n(2n-1)}{3} + 2n + 1\right) \\
&= (2n+1)\left(\frac{2n^2 - n + 6n + 3}{3}\right) \\
&= (2n+1)\left(\frac{2n^2 + 5n + 3}{3}\right) \\
&= \frac{(2n+1)(n+1)(2n+3)}{3} \\
&= \frac{(n+1)(2(n+1)-1)(2(n+1)+1)}{3}.
\end{aligned}$$

Thus the statement is true for $n+1$. By induction, it holds for all $n \geq 1$.

2. We prove the statement by induction on n.

   For $n = 1$,

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}.$$

   So the statement is true for $n = 1$.

   Now assume that the statement is true for some $n \geq 1$, that is,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

   Then

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}.
\end{aligned}$$

   This is exactly the desired formula with $n+1$ in place of n. By induction, the statement holds for all $n \geq 1$.

3. We prove the statement by induction on n.

   For $n = 1$,
$$\sum_{i=1}^{1} i(i!) = 1(1!) = 1 = 2! - 1.$$
   So the statement is true for $n = 1$.

   Now assume that the statement is true for some $n \geq 1$, that is,
$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$
   Then
$$\begin{aligned}\sum_{i=1}^{n+1} i(i!) &= \sum_{i=1}^{n} i(i!) + (n+1)((n+1)!) \\ &= (n+1)! - 1 + (n+1)((n+1)!) \\ &= (n+2)(n+1)! - 1 \\ &= (n+2)! - 1.\end{aligned}$$
   This is the desired formula with $n+1$ in place of n. By induction, the statement holds for all $n \geq 1$.

Problem 2. *(Only for those familiar with complex numbers)* Prove DeMoivre's theorem
$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Answer.

We prove the theorem by induction on $n \geq 1$.
For $n = 1$, both sides are equal to $\cos\theta + i\sin\theta$, so the statement is true.
Now assume the statement is true for some $n \geq 1$, that is,
$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$
Then
$$\begin{aligned}(\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) \\ &= (\cos(n\theta) + i\sin(n\theta))(\cos\theta + i\sin\theta) \\ &= \quad \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta \\ &\quad + i\big(\sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta\big) \\ &= \cos((n+1)\theta) + i\sin((n+1)\theta),\end{aligned}$$
where we used the addition formulas for sine and cosine[a]. Therefore the statement is true for $n+1$. By induction, DeMoivre's theorem holds for all $n \geq 1$.

[a]Recall $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha)$ and $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

Problem 3. Prove that for all natural numbers n if $n > 3$ then $2^n < n!$.

Answer.

We prove by induction that $2^n < n!$ for all $n \geq 4$.
For $n = 4$,

$$2^4 = 16 < 24 = 4!.$$

So the statement is true for $n = 4$.
Now assume that the statement is true for some $k \geq 4$, that is,

$$2^k < k!.$$

Then

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot k!.$$

Since $k \geq 4$, we have $k + 1 > 2$, and hence

$$2 \cdot k! < (k+1)k! = (k+1)!.$$

Therefore

$$2^{k+1} < (k+1)!.$$

By induction, $2^n < n!$ for all $n \geq 4$, which is the same as saying that for all natural numbers n, if $n > 3$, then $2^n < n!$.

Problem 4. A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n - 1$ moves are required to assemble a puzzle with n pieces.

Answer.

We prove the statement by strong induction on n.
For $n = 1$, the puzzle already consists of one piece, so no move is needed. Since $1 - 1 = 0$, the statement is true.
Now let $n \geq 2$, and assume that for every $m < n$, any puzzle with m pieces requires exactly $m - 1$ moves to assemble.[a]
Consider a puzzle with n pieces. Look at the last move. In this last move, two blocks are joined together to form the complete puzzle. Suppose these two blocks have sizes a and b, where

$$a + b = n$$

and $a, b \geq 1$. Since $n \geq 2$, both a and b are smaller than n. By the strong induction hypothesis, the block of size a required exactly $a - 1$ moves to assemble, and the block of size b required exactly $b - 1$ moves to assemble.
The final move joins these two blocks. Therefore the total number of moves is

$$(a - 1) + (b - 1) + 1 = a + b - 1 = n - 1.$$

Thus any puzzle with n pieces requires exactly $n-1$ moves. By strong induction, the statement holds for all $n \geq 1$.

[a] In general, in a **strong** induction hypothesis, we assume **all** statements $P(1), P(2), \ldots, P(k)$ are true for some k.

Problem 5. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps.

1. Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of the proof.
2. What is inductive hypothesis of the proof?
3. What do you need to prove in the inductive step?
4. Complete the inductive step for $k \geq 10$.

Answer.

This proof is meant to show that every postage of n cents can be formed using only 3-cent and 5-cent stamps, for all $n \geq 8$.

1. We check the three basis cases:
$$8 = 3 + 5,$$
so $P(8)$ is true. Also,
$$9 = 3 + 3 + 3,$$
so $P(9)$ is true. Finally,
$$10 = 5 + 5,$$
so $P(10)$ is true.

2. The strong induction hypothesis is: assume that for some $k \geq 10$, all of
$$P(8), P(9), \ldots, P(k)$$
are true. In other words, assume that every postage amount from 8 cents up to k cents can be formed using only 3-cent and 5-cent stamps.

3. In the inductive step, we need to prove that $P(k+1)$ is true. That is, we need to show that a postage of $k+1$ cents can be formed using only 3-cent and 5-cent stamps.

4. Assume $k \geq 10$ and assume the strong induction hypothesis, so that $P(8), P(9), \ldots, P(k)$ are all true.

   Since $k \geq 10$, we have
$$k - 2 \geq 8.$$
   Also, $k - 2 \leq k$, so $P(k-2)$ is true by the induction hypothesis. Therefore, a postage of $k-2$ cents can be formed using only 3-cent and 5-cent stamps.

   Now add one more 3-cent stamp. This gives a postage of
$$(k-2) + 3 = k + 1$$
   cents. Hence $P(k+1)$ is true.

   Therefore, by strong induction, every postage amount $n \geq 8$ can be formed using only 3-cent and 5-cent stamps.