Solutions to Week 10 Tutorial — MACM 101 (Spring 2025)

Problem 1. A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with n pieces.

Answer.

We use strong induction on n.
For $n=1$, the puzzle is already assembled, so no moves are required. This is $1-1=0$ moves. Now suppose that the result holds for every puzzle with fewer than n pieces. Consider the final move in assembling a puzzle with n pieces. This move joins two completed blocks, one containing k pieces and the other containing $n-k$ pieces, where $1 \leq k < n$. By the induction hypothesis, assembling these two blocks requires $k-1$ and $n-k-1$ moves, respectively. The final move then joins them, so the total number of moves is

$$(k-1)+(n-k-1)+1=n-1.$$

Therefore every puzzle with n pieces requires exactly $n-1$ moves, regardless of the order in which the moves are made.

Problem 2. Give a recursive definition of $P_m(n)$, the product of the integer m and the non-negative integer n.

Answer.

Fix an integer m. Define $P_m(n)$ recursively by

$$P_m(0)=0$$

and, for every non-negative integer n,

$$P_m(n+1)=P_m(n)+m.$$

Thus, each step adds one more copy of m, so $P_m(n)=mn$.

Problem 3. Let $F(n)$ be the nth Fibonacci number. Prove that

$$F(1)+F(2)+\cdots+F(n)=F(n+2)-1.$$

Answer.

We use induction on n, with $F(1)=F(2)=1$.
For $n=1$,

$$F(1)=1=F(3)-1.$$

Now suppose that

$$F(1)+F(2)+\cdots+F(k)=F(k+2)-1.$$

Then

$$\begin{aligned} F(1)+F(2)+\cdots+F(k)+F(k+1) &= F(k+2)-1+F(k+1) \\ &= F(k+3)-1, \end{aligned}$$

since $F(k+3) = F(k+2) + F(k+1)$. Therefore the identity holds for every positive integer n.

Problem 4. A complete binary tree is a graph defined through the following recursive definition.

Basis step: A single vertex is a complete binary tree.

Inductive step: If T_1 and T_2 are disjoint complete binary trees with roots r_1 and r_2, respectively, then the graph formed by starting with a root r, and adding an edge from r to each of the vertices r_1 and r_2, is also a complete binary tree.

Draw all complete binary trees that can be obtained by applying the inductive step at most three times (complete binary trees of level 3). Use structural induction to show that $n(T) \geq 2h(T) + 1$, where $n(T)$ denotes the number of vertices in T, and $h(T)$ denotes the height of T.

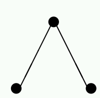
Answer.

Comment: *In the usual convention for binary trees, the left and right children are distinguished. Under this convention, a tree and its mirror image are different, so there are nine trees in total. If the trees are viewed only as graphs, without distinguishing left from right, then mirror images are isomorphic and there are only five trees.*

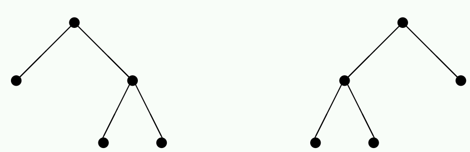
The following are all the trees when left and right children are distinguished. The diagrams are grouped according to the number of times the inductive step is applied. The number of calls to the inductive step is the number of non-leaf vertices.

Zero applications

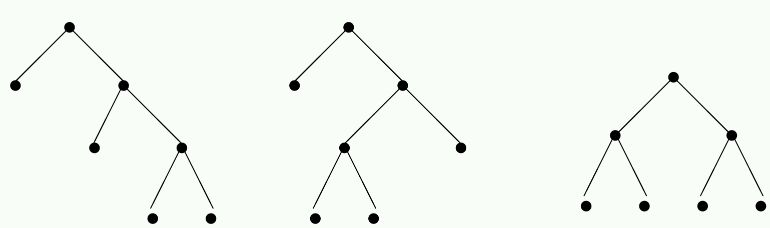
One application

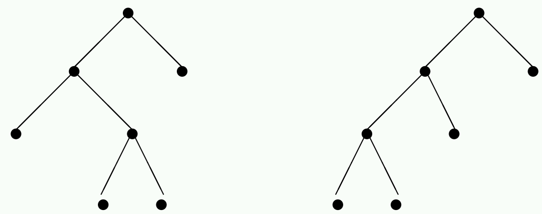


Two applications



Three applications





We now prove the inequality by structural induction.
For the basis tree, T consists of a single vertex. Therefore, $n(T) = 1$ and $h(T) = 0$, so

$$n(T) = 1 = 2h(T) + 1.$$

For the inductive step, suppose that T is formed from two complete binary trees T_1 and T_2. Assume that

$$n(T_1) \geq 2h(T_1) + 1 \qquad \text{and} \qquad n(T_2) \geq 2h(T_2) + 1.$$

Adding the new root gives

$$n(T) = n(T_1) + n(T_2) + 1,$$

while the height of the new tree is

$$h(T) = 1 + \max\{h(T_1), h(T_2)\}.$$

Consequently,

$$\begin{aligned} n(T) &= n(T_1) + n(T_2) + 1 \\ &\geq 2h(T_1) + 2h(T_2) + 3 \\ &\geq 2\max\{h(T_1), h(T_2)\} + 3 \\ &= 2h(T) + 1. \end{aligned}$$

Thus, by structural induction,

$$n(T) \geq 2h(T) + 1$$

for every complete binary tree T.

Problem 5. Determine the quotient q and the remainder r for each of the following, where a is the dividend and b is the divisor.

(a) $a = 123$, $b = 17$; (b) $a = -115$, $b = 12$;
(c) $a = 0$, $b = 42$; (d) $a = 434$, $b = 31$.

Answer.

Tip: *Remainder is always non-negative.*

We write $a = bq + r$, where $0 \leq r < b$. In each case, the quotient and remainder are as follows:

$$
\begin{array}{llll}
\text{(a)} & 123 = 17(7) + 4, & q = 7, & r = 4; \\
\text{(b)} & -115 = 12(-10) + 5, & q = -10, & r = 5; \\
\text{(c)} & 0 = 42(0) + 0, & q = 0, & r = 0; \\
\text{(d)} & 434 = 31(14) + 0, & q = 14, & r = 0.
\end{array}
$$

Problem 6. Write each of the following base-10 numbers in base 2 and base 16.

(a) 137; (b) 6243; (c) 12345.

Answer.

To convert to base 2, repeatedly divide by 2 and record the remainders. The binary expansion is obtained by reading the remainders from bottom to top.

(a) For 137, the repeated divisions are

$$
\begin{aligned}
137 &= 2(68) + 1, \\
68 &= 2(34) + 0, \\
34 &= 2(17) + 0, \\
17 &= 2(8) + 1, \\
8 &= 2(4) + 0, \\
4 &= 2(2) + 0, \\
2 &= 2(1) + 0, \\
1 &= 2(0) + 1.
\end{aligned}
$$

Thus

$$137 = (10001001)_2.$$

For base 16,

$$137 = 16(8) + 9,$$

so $137 = (89)_{16}$.

(b) For 6243, the repeated divisions are

$$
\begin{aligned}
6243 &= 2(3121) + 1, \\
3121 &= 2(1560) + 1, \\
1560 &= 2(780) + 0, \\
780 &= 2(390) + 0, \\
390 &= 2(195) + 0, \\
195 &= 2(97) + 1, \\
97 &= 2(48) + 1, \\
48 &= 2(24) + 0, \\
24 &= 2(12) + 0, \\
12 &= 2(6) + 0, \\
6 &= 2(3) + 0, \\
3 &= 2(1) + 1, \\
1 &= 2(0) + 1.
\end{aligned}
$$

Thus

$$6243 = (1100001100011)_2.$$

For base 16,

$$\begin{aligned}
6243 &= 16(390) + 3, \\
390 &= 16(24) + 6, \\
24 &= 16(1) + 8, \\
1 &= 16(0) + 1,
\end{aligned}$$

so $6243 = (1863)_{16}$.

Tip (base-16 $\rightarrow$ base-2): You can directly convert the base-16 representation to a base-2 representation. Each digit in base 16 can be represented with a 4-digit binary number. So simply write each digit of the base-16 representation as a 4-digit binary representation by adding leading zeros if you have to. Therefore, given that $6243 = (1863)_{16}$, we find its binary representation:

$$\underbrace{1}_{0001}\ \underbrace{8}_{1000}\ \underbrace{6}_{0110}\ \underbrace{3}_{0011}.$$

Finally you can get rid of the leading zeros of the most significant digit (which is 1 in this case, and is marked in red).

This conversion works nicely because the base we started with is a power of the base we want to convert to ($16 = 2^4$).

(c) For 12345, the repeated divisions are

$$\begin{aligned}
12345 &= 2(6172) + 1, \\
6172 &= 2(3086) + 0, \\
3086 &= 2(1543) + 0, \\
1543 &= 2(771) + 1, \\
771 &= 2(385) + 1, \\
385 &= 2(192) + 1, \\
192 &= 2(96) + 0, \\
96 &= 2(48) + 0, \\
48 &= 2(24) + 0, \\
24 &= 2(12) + 0, \\
12 &= 2(6) + 0, \\
6 &= 2(3) + 0, \\
3 &= 2(1) + 1, \\
1 &= 2(0) + 1.
\end{aligned}$$

Thus

$$12345 = (11000000111001)_2.$$

For base 16,

$$\begin{aligned} 12345 &= 16(771) + 9, \\ 771 &= 16(48) + 3, \\ 48 &= 16(3) + 0, \\ 3 &= 16(0) + 3, \end{aligned}$$

so $12345 = (3039)_{16}$.

Problem 7. Convert each of the following binary numbers to base 10.

(a) 11001110; (b) 00110001.

Answer.

(a)

$$\begin{aligned} (11001110)_2 &= 2^7 + 2^6 + 2^3 + 2^2 + 2^1 \\ &= 128 + 64 + 8 + 4 + 2 \\ &= 206. \end{aligned}$$

(b)

$$\begin{aligned} (00110001)_2 &= 2^5 + 2^4 + 2^0 \\ &= 32 + 16 + 1 \\ &= 49. \end{aligned}$$