Question 1 [3 points]: A universal classical gate

The NAND gate is a classical gate with the following truth table:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>NAND(x, y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

1. Show that the NOT gate can be implemented with NAND gates and FANOUT. You may draw a circuit or simply give the algebraic expression.

2. Show that the gate set \{NAND, FANOUT\} is universal for classical computation by giving implementations of each gate in the universal gate set \{AND, OR, NOT, FANOUT\}.

Question 2 [6 points]: Dirac notation

Let \( |\psi\rangle = \frac{1}{\sqrt{3}} |0\rangle + \frac{i}{\sqrt{3}} |1\rangle + \frac{-1}{\sqrt{3}} |2\rangle \), \( |\phi\rangle = \frac{1}{\sqrt{2}} |1\rangle + \frac{-i}{\sqrt{2}} |2\rangle \) be two states of a qutrit (i.e. a three-level or three-dimensional system).

1. Give the explicit column vectors of \( |\psi\rangle \) and \( |\phi\rangle \)

2. Calculate the following:
   - \( \langle \psi | \psi \rangle \)
   - \( \langle \phi | \phi \rangle \)
   - \( \langle \psi | \phi \rangle \)
   - \( |\psi\rangle \langle \phi | \)
   - \( |\psi\rangle \otimes |\phi \rangle \)

3. Is the vector \( |\psi\rangle + |\phi\rangle \) a unit vector? If not, normalize it to get a unit vector.
Question 3 [4 points]: Gates and measurement

Suppose we have a qubit initially in the state \( \frac{1}{\sqrt{2}} |0\rangle + \frac{e^{i\theta}}{\sqrt{2}} |1\rangle \) for some \( \theta \in \mathbb{R} \).

1. Calculate the probabilities of receiving result “0” or “1” if the qubit is measured.

2. Recall the definition of the Hadamard gate, which has the vectors |+\rangle and |−\rangle as its columns:

\[
H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
\]

If we first apply the Hadamard gate to the initial state \( \frac{1}{\sqrt{2}} |0\rangle + \frac{e^{i\theta}}{\sqrt{2}} |1\rangle \) and then measure, what are the probabilities of receiving the “0” and “1” results as a function of \( \theta \)?

Note: this is the same thing as measuring the initial state in the |+\rangle, |−\rangle basis.

Question 4 [5 points]: Eigenvectors

Recall that an eigenvector of a matrix \( A \) is a vector |v\rangle such that \( A|v\rangle = \lambda|v\rangle \) for some scalar eigenvalue \( \lambda \).

1. Let \( Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \). Find two unit vectors |+Y\rangle, |−Y\rangle such that

\[
Y|+Y\rangle = |+Y\rangle \\
Y|−Y\rangle = −|−Y\rangle
\]

2. Let \( U \) be the 2 by 2 matrix with columns |+Y\rangle and |−Y\rangle. Is \( U \) unitary?

3. Calculate \( U^\dagger YU \).