Question 1 [3 points]: Optimal angles for the Zeno effect

In class we saw that we can drag a state from the $|0\rangle$ state to the $|1\rangle$ state by performing measurements in rotated bases. Given a basis $\mathcal{B} = \{|A\rangle, |B\rangle\}$ define the basis $\mathcal{B}$ rotated by an angle $\theta$ to be $\{\cos(\theta)|A\rangle + \sin(\theta)|B\rangle, -\sin(\theta)|A\rangle + \cos(\theta)|B\rangle\}$. Observe that this basis is in fact orthonormal via the identity $\cos^2(\theta) + \sin^2(\theta) = 1$.

1. Show that rotating the basis twice by $\theta$ is the same as rotating once by an angle of $2\theta$
2. Calculate the angle $\theta$ and number of measurements needed to reach the $|1\rangle$ state with success probability at least $p$ for some positive real number $p$ close to 1.

Note: Assume that $\sin^2(x) = x^2$ when $x$ is close to 0. You will likely need to use the union bound, i.e. Boole’s inequality

\[ pr(A \lor B \lor C \lor \cdots) \leq pr(A) + pr(B) + pr(C) + \cdots \]

Question 2 [2 points]: State discrimination

Using computational basis measurement, $H$ gates, and phase gates

\[ P(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \]

where $\theta$ can be any real number, give a protocol to distinguish with 100% accuracy between the states

\[ |\psi\rangle = \frac{1}{\sqrt{2}}(e^{i\pi/4}|0\rangle + |1\rangle), \quad |\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i3\pi/4}|1\rangle) \]

Question 3 [4 points]: Pauli operators

Recall the definition of the $I$, $X$, $Z$, and $Y$ gates:

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \]

These are known as the Pauli matrices or gates.
1. Compute the matrices $X \otimes Z$ and $Z \otimes X$

2. Show that the non-identity Pauli matrices anti-commute: that is, $UV = -VU$ for every pair of $X$, $Y$, and $Z$ matrices where $U \neq V$

3. Show that the Pauli matrices $I, X, Z, Y$ are linearly independent

4. Show that the Pauli matrices form a basis for the space of $2 \times 2$ complex-valued matrices.

**Question 4 [2 points]: Entanglement**

Prove that the controlled-$Z$ gate

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is entangling. Do so by giving an explicit two-qubit (unentangled) state $|\psi\rangle \otimes |\phi\rangle$ and showing that $CZ(|\psi\rangle \otimes |\phi\rangle)$ is entangled.

**Question 5 [2 points]: Partial measurement**

Let

$$|\psi\rangle = \frac{i\sqrt{2}}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}\sqrt{2}}|01\rangle + \frac{\sqrt{2}}{2\sqrt{3}}|10\rangle.$$ 

Calculate the probabilities of measuring 0 or 1 in the first qubit, and the resulting normalized state vector in either case.

**Question 6 [9 points]: Non-local games**

In this question, we’re going to examine another non-local game involving 3-parties, or 3 qubits. First let $|\psi\rangle = \frac{1}{2}(|000\rangle - |110\rangle - |011\rangle - |101\rangle)$

1. Give a 3-qubit circuit $U$ consisting of $X$, $H$, and $CNOT$ gates such that

$$U \left( \frac{1}{\sqrt{2}}|000\rangle - \frac{1}{\sqrt{2}}|111\rangle \right) = |\psi\rangle.$$

2. Show that a partial measurement of any qubit in the $|\psi\rangle$ state leaves an entangled state in the remaining 2 qubits.

3. Compute the parity $a \oplus b \oplus c = a + b + c \mod 2$ of the measurement results if

   (a) All qubits are measured in the $\{0, 1\}$ basis.

   (b) Qubits 0 and 1 are measured in the $\{|+, -\rangle\}$ basis and qubit 2 in the $\{0, 1\}$ basis.

   (c) Qubits 0 and 2 are measured in the $\{|+, -\rangle\}$ basis and qubit 1 in the $\{0, 1\}$ basis.

   (d) Qubits 1 and 2 are measured in the $\{|+, -\rangle\}$ basis and qubit 0 in the $\{0, 1\}$ basis.
Note: in the $\{|+\rangle,\{-\rangle\}$ basis, we consider the result of measuring “+” to be 0 and the result of measuring “-” to be 1.

4. Denote the measurement result of qubit $i$ in the $\{0,1\}$ basis by $a_i$, and in the $\{|+\rangle,\{-\rangle\}$ basis by $b_i$. Is it possible that each $a_i$ and $b_i$ has a **pre-determined value** independent of which basis the other qubits are measured in? Give a convincing argument for your answer.

5. Give a quantum strategy (i.e. a strategy where involving a shared pre-entangled state) for a 3-player game where Alice, Bob, and Charlie are each given one bit $x$, $y$, and $z$ respectively, and have to return a single bit $a$, $b$, $c$ respectively. They win if $a \oplus b \oplus c = x \lor y \lor z$. **Jan 29th update:** you should assume that $x \oplus y \oplus z = 0$ and your strategy should win the game with 100% probability.

Hint: use the state $|\psi\rangle$ from the first part of this question as the initial shared state.