SIMON FRASER UNIVERSITY
School of Computing Science

CMPT 476/981– MIDTERM EXAM
Introduction to Quantum Algorithms

Instructor: Matt Amy
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Name: ________________________________
Student Number: ________________________________

Instructions: 

• 1 double-sided sheet of 8.5x11” paper is permitted as a cheat-sheet
• A non-programmable calculator is permitted
• No other aids are permitted
• Print your full name and student ID number in the space above
• There are 10 pages including this cover page and 8 questions
• The total number of points is 46.
• You will have 110 minutes
• Good luck!

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1
1. (10 points) Short answers, 1 point each
   
   (a) What is the dimension of the state space of \( n \) qubits?
       \[ 2^n \]
   
   (b) What is the definition of a unitary operator (you do not need to define the dagger (\( \cdot \dagger \))
       \[ U \dagger = U^{-1} \]
   
   (c) What is the maximum number of dimensions a single particle’s quantum state can have?
       There is no maximum number
   
   (d) What is the probability of measuring \( |0\rangle \) in the state \( a|0\rangle + b|1\rangle + c|2\rangle \)?
       \[ |a|^2 \]
   
   (e) Write the state \( a|0\rangle + b|1\rangle + c|2\rangle \) as a vector.
       \[ \begin{bmatrix} a \\ b \\ c \end{bmatrix}^T \]
   
   (f) Give one way in which quantum computation is different from probabilistic computation.
       States in quantum computation can have ”negative probabilities”
   
   (g) Normalize the vector \( \sqrt{5}|0\rangle + \sqrt{-11}|1\rangle \)
       \[ \frac{\sqrt{5}}{4}|0\rangle + \frac{\sqrt{-11}}{4}|1\rangle \]
   
   (h) Complete the expression:
       \[ e^{i\theta} = \cos \theta + i \sin \theta \]
   
   (i) Complete the expression:
       \[ \text{Tr} \left( \begin{bmatrix} 3 & 5 & 1 \\ 0 & 4 & 9 \\ 5 & 5 & 5 \end{bmatrix} \right) = 12 \]
   
   (j) Give one way a (quantum) controlled gate \( c \) – \( U \) is different from a classically controlled
       gate \( U^x \) — i.e. applying a gate depending on the value of a classical bit \( x \in \{0, 1\} \).
       A quantum controlled gate can be applied with a superposition of different control values
2. (a) (3 points) Calculate the probabilities of obtaining each result when measuring the state

\[ |\psi\rangle = \frac{3}{5}|0\rangle + \frac{-4i}{5}|1\rangle \]

in the basis

\[ \{|A\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \}, \quad |B\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \} \]

**Probability of measuring A:**

\[ |\langle A|\psi\rangle|^2 = |(\frac{1}{\sqrt{2}}|0\rangle + \frac{-i}{\sqrt{2}}|1\rangle)(\frac{3}{5}|0\rangle + \frac{-4i}{5}|1\rangle)|^2 \]

\[ = \left| \frac{3}{5\sqrt{2}} \langle 0|0\rangle + \frac{-4i}{5\sqrt{2}} \langle 0|1\rangle + \frac{-3i}{5\sqrt{2}} \langle 1|0\rangle + \frac{-4}{5\sqrt{2}} \langle 1|1\rangle \right|^2 \]

\[ = \left| \frac{-1}{5\sqrt{2}} \right|^2 \]

\[ = \frac{1}{50} \]

**Probability of measuring B:**

\[ |\langle B|\psi\rangle|^2 = |\|^2 = \left| \frac{-3i}{5\sqrt{2}} \langle 0|0\rangle + \frac{-4i}{5\sqrt{2}} \langle 1|1\rangle \right|^2 = \left| \frac{-7i}{5\sqrt{2}} \right|^2 = \frac{49}{50} \]

(b) (4 points) Calculate the probabilities and give the corresponding final state if the first qubit of the state

\[ |\psi\rangle = \frac{1}{5}|00\rangle + \frac{\sqrt{3}}{5}|01\rangle + \frac{1}{\sqrt{5}}|10\rangle + \frac{4}{5}|11\rangle \]

is measured in the computational basis.

If \(|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\) then

\[ |\psi\rangle = \sqrt{|a|^2 + |b|^2}|0\rangle \otimes \left( \frac{a|0\rangle + b|1\rangle}{\sqrt{|a|^2 + |b|^2}} \right) + \sqrt{|c|^2 + |d|^2}|1\rangle \otimes \left( \frac{c|0\rangle + d|1\rangle}{\sqrt{|c|^2 + |d|^2}} \right) \]

Then the partial measurement rule applies with \(p(0) = |a|^2 + |b|^2\) and \(p(1) = |c|^2 + |d|^2\).

**Measurement result 0:**

- Probability is \(p(0) = \frac{1}{5}|^2 + \frac{\sqrt{3}}{5}|^2 = \frac{4}{25} \)
- Final state is \( \frac{\frac{1}{5}|00\rangle + \frac{\sqrt{3}}{5}|01\rangle}{\sqrt{p(0)}} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}}|00\rangle + \frac{\sqrt{3}}{5}|01\rangle \right) = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|01\rangle \)

**Measurement result 1:**

- Probability is \(p(1) = \frac{1}{\sqrt{5}}|^2 + \frac{4}{5}|^2 = \frac{21}{25} \)
- Final state is \( \frac{\frac{1}{\sqrt{5}}|10\rangle + \frac{2}{5}|11\rangle}{\sqrt{p(1)}} = \frac{5}{\sqrt{21}} \left( \frac{1}{\sqrt{5}}|10\rangle + \frac{2}{5}|11\rangle \right) = \frac{\sqrt{5}}{\sqrt{21}}|10\rangle + \frac{4}{\sqrt{21}}|11\rangle \)
3. (a) (2 points) Write the following 3-qubit state as a linear combination over the 3-qubit computational (binary) basis:

\[
\frac{1}{2} \left( |001\rangle + i|011\rangle - |100\rangle - i|110\rangle \right)
\]

(b) (3 points) Write the following 2-qubit operator as a linear combination over the computational basis of 2-qubit operators \{\langle lk| |i,j\rangle\} \in \{0,1\}\}

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 & -i\sqrt{2} \\
0 & 1 & i & 0 \\
i\sqrt{2} & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\frac{1}{\sqrt{2}} \left( -i\sqrt{2} |00\rangle \langle 11| + |01\rangle \langle 01| + i|01\rangle \langle 10| + i|10\rangle \langle 01| + |10\rangle \langle 10| + i\sqrt{2} |11\rangle \langle 00| \right)
\]

(c) (2 points) To combat noise and decoherence, we often encode a logical qubit inside a subspace of a larger Hilbert space. Suppose we use the subspace span\{(00), |11\}\} of \mathbb{C}^2 \otimes \mathbb{C}^2 to encode one logical qubit, where the “0” state is taken to be |00\rangle and the “1” state is |11\rangle. What 1 qubit gate (i.e. a 2 \times 2 unitary transformation) does the operator in the previous part perform on this subspace?

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -i\sqrt{2} \\
i\sqrt{2} & 0
\end{pmatrix} \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} = Y
\]
4. (a) (2 points) Let
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \\ i & j \end{bmatrix}.
\]
Write down the matrix \(A \otimes B\).

\[
\begin{bmatrix}
ae & af & be & bf \\
ag & ah & bg & bh \\
ai & aj & bi & bj \\
ce & cf & de & df \\
cg & ch & dg & dh \\
\end{bmatrix}
\]

(b) (4 points) Let \(\mathbb{C}^{m \times n}\) denote the space of complex-valued matrices with \(m\) rows and \(n\) columns, and define the following constants:
\[
|\psi\rangle \in \mathbb{C}^{2} = \mathbb{C}^{2 \times 1}, \quad A \in \mathbb{C}^{2 \times 2}, \quad E \in \mathbb{C}^{4 \times 8} \\
|\phi\rangle \in \mathbb{C}^{8} = \mathbb{C}^{8 \times 1}, \quad B \in \mathbb{C}^{8 \times 8}, \quad I \in \mathbb{C}^{2 \times 2} \\
|\Delta\rangle \in \mathbb{C}^{4} = \mathbb{C}^{4 \times 1}, \quad C \in \mathbb{C}^{4 \times 4} \\
|\zeta\rangle \in \mathbb{C}^{4} = \mathbb{C}^{4 \times 1}, \quad D \in \mathbb{C}^{4 \times 4}
\]
Correctly parenthesize the expression below to make it well-formed, keeping in mind that
- \(A + B\) is well formed if and only if the dimensions of \(A\) and \(B\) are equal, and
- \(AB\) is well-formed if and only if the columns of \(A\) equal the rows of \(B\).

**Hint: work from right to left**

\[
(\langle \zeta | \otimes I) \cdot (E \otimes I) \cdot ((C + |\Delta\rangle \langle \Delta|) \otimes D) \cdot (B \otimes I) \cdot (|\phi\rangle \otimes (A \cdot |\psi\rangle))
\]

(c) (1 point (bonus)) Draw the expression in part (b) as a circuit diagram.
5. (5 points) Calculate the final state of the circuit below in the computational basis.

\[
|000\rangle \xrightarrow{I \otimes H \otimes I} \frac{1}{\sqrt{2}} (|000\rangle + |010\rangle)
\]

\[
I \otimes I \otimes H \xrightarrow{\frac{1}{2}} (|000\rangle + |001\rangle + |010\rangle + |011\rangle)
\]

\[
X \otimes I \otimes I \xrightarrow{\frac{1}{2}} (|100\rangle + |101\rangle + |110\rangle + |111\rangle)
\]

\[
\text{CNOT}_{2,3} \xrightarrow{\frac{1}{2}} (|100\rangle + |101\rangle + |111\rangle + |110\rangle)
\]

\[
\text{CNOT}_{3,1} \xrightarrow{\frac{1}{2}} (|100\rangle + |001\rangle + |011\rangle + |110\rangle)
\]

\[
\text{CNOT}_{1,2} \xrightarrow{\frac{1}{2}} (|110\rangle + |001\rangle + |011\rangle + |100\rangle)
\]
6. (2 points) Recall that in the Bloch sphere, a qubit has state $|\psi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\varphi} \sin(\frac{\theta}{2})|1\rangle$ where $\theta$ is the angle the state makes with the positive $z$-axis and $\varphi$ the angle it makes with the positive $x$-axis.

Implement a transformation that maps the $|0\rangle$ state to any point $\cos(\frac{\theta}{2})|0\rangle + e^{i\varphi} \sin(\frac{\theta}{2})|1\rangle$ on the Bloch sphere using rotations around the $x$-, $y$-, and/or $z$-axes. Recall that the corresponding rotation matrices are defined as

$$
R_x(\theta) = \begin{bmatrix}
\cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\
-i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2})
\end{bmatrix}, \quad
R_y(\theta) = \begin{bmatrix}
\cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\
\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2})
\end{bmatrix}, \quad
R_z(\theta) = \begin{bmatrix}
e^{-i\theta/2} & 0 \\
0 & e^{i\theta/2}
\end{bmatrix}
$$

We can arrive at the point $|\psi\rangle$ by first rotating around the $y$ axis $\theta$ degrees, then around the $z$ axis $\varphi$ degrees. In particular,

$$
R_z(\varphi)R_y(\theta)|0\rangle = R_z(\varphi)(\cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle)
= e^{-i\varphi/2} \cos(\theta/2)|0\rangle + e^{i\varphi/2} \sin(\theta/2)|1\rangle
= e^{-i\varphi/2}(\cos(\theta/2)|0\rangle + e^{i\varphi} \sin(\theta/2)|1\rangle)
= |\psi\rangle
$$

where the last equality uses global phase invariance.
7. (4 points) Using $CNOT$ and $H$ gates and computational-basis measurement, give a procedure to distinguish with 100% accuracy between the following states

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle), \quad |\phi\rangle = \frac{1}{\sqrt{2}}(|1100\rangle - |0011\rangle)$$

Apply the following circuit $U$:

and return $|\psi\rangle$ if the measurement result is $|0\rangle$, or $|\phi\rangle$ otherwise.

**proof:** Note that the first 3 $CNOT$ gates send $|1100\rangle$ to $|1011\rangle$ and $|0011\rangle$ to $|0011\rangle$. Hence the $CNOT$ gates map $|\psi\rangle$ to $\frac{1}{\sqrt{2}}(|1101\rangle + |0011\rangle) = |+\rangle \otimes |011\rangle$ and $|\phi\rangle$ to $\frac{1}{\sqrt{2}}(|1101\rangle - |0011\rangle) = -|-\rangle \otimes |011\rangle$. Therefore, after the final $H$ gate the first qubit is in the state $|0\rangle$ if the original state was $|\psi\rangle$, and $|1\rangle$ if the original state was $|\phi\rangle$. 
8. Suppose Alice and Bob share an EPR pair $|\beta_{00}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$. Suppose Bob applies a unitary $U$ to his qubit and then keeps it in storage.

(a) (4 points) Suppose a year has passed and Alice creates some qubit in the state $|\psi\rangle$. She then measures this qubit with her half of the EPR pair in the Bell basis to teleport it to Bob, and obtains measurement result $\beta_{00}$. What is the resulting state of Bob’s qubit? If it helps, recall the definition of the Bell basis:

\[
|\beta_{00}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \quad |\beta_{01}\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \\
|\beta_{10}\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \quad |\beta_{11}\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)
\]

First we write the state of all 3 qubits with Alice’s qubits in the Bell basis. Let $|\psi\rangle = a|0\rangle + b|1\rangle$.

\[
|\psi\rangle \otimes (I \otimes U)|\beta_{00}\rangle = (a|0\rangle + b|1\rangle) \otimes \frac{1}{\sqrt{2}} ((0) \otimes U|0\rangle + |1\rangle \otimes U|1\rangle) \\
= \frac{1}{\sqrt{2}} (a|00\rangle \otimes U|0\rangle + a|01\rangle \otimes U|1\rangle + b|10\rangle \otimes U|0\rangle + b|11\rangle \otimes U|1\rangle) \\
= \frac{1}{2} (a(|\beta_{00}\rangle + |\beta_{10}\rangle) \otimes U|0\rangle + a(|\beta_{01}\rangle + |\beta_{11}\rangle) \otimes U|1\rangle) \\
+ b(|\beta_{01}\rangle - |\beta_{11}\rangle) \otimes U|0\rangle + b(|\beta_{00}\rangle - |\beta_{10}\rangle) \otimes U|1\rangle) \\
= \frac{1}{2} (|\beta_{00}\rangle \otimes (aU|0\rangle + bU|1\rangle) + |\beta_{01}\rangle \otimes (aU|1\rangle + bU|0\rangle) \\
+ |\beta_{10}\rangle \otimes (aU|0\rangle - bU|1\rangle) + |\beta_{11}\rangle \otimes (aU|1\rangle - bU|0\rangle))
\]

So if Alice obtains measurement result $|\beta_{00}\rangle$ when measuring in the Bell basis, Bob has the state $aU|0\rangle + bU|1\rangle = U(a|0\rangle + b|1\rangle) = U|\psi\rangle$.

(b) (1 point) Argue whether or not the above could be considered to be a violation of causality — the notion that causes and effect occur in the order in which they happen.

Yes, this could be viewed as a violation of causality, because it appears that the gate $U$ which Bob applied to his qubit before $|\psi\rangle$ existed is actually applied after Alice creates the state $|\psi\rangle$. Of course, this must be preposterous and can’t actually be what happens. What’s more likely is that Bob’s state $|\phi\rangle$ after teleportation is related to Alice’s state $|\psi\rangle$ by the relation $|\phi\rangle = U|\psi\rangle$. 

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