Questions 1-3 will be about a mini language $L$ for controlling robots that travel along a line. $L$ is defined by the following syntax:

$$s \ ::= \ \text{move } n \mid \text{flip} \mid \text{skip} \mid s ; s \mid n \in \mathbb{R}$$

1 **Design Semantics**

Design a semantics for the language by writing an inductive relation:

$$s \vdash (x, b) \rightsquigarrow (x', b') \subseteq \mathbb{L} \times \mathbb{R} \times \mathbb{B}$$

where $\mathbb{B} = \{\bot, \top\}$ is the set of boolean values ($\bot$ represents false and $\top$ represents true) and $\mathbb{R}$ is the set of all real numbers. These semantics should be expressed as an inductive relation defined by inference rules.

In this relation, $s \vdash (x, b) \rightsquigarrow (x', b')$ should represent that, if a robot starts at position $x$, facing to the right if $b = \top$ and facing to the left if $b = \bot$, then after running the program described by $s$, the robot ends at position $x'$ and faces to the right if $b' = \top$ and faces to the left if $b' = \bot$.

In this language, $\text{move } n$ represents that the program moves “forward” by $n$ steps (in other words, $n$ is added to its position if the robot is facing to the right, and $n$ is subtracted from its position if the robot is facing to the left).

In this language, $\text{flip}$ represents that the robot changes direction. In other words, if the robots position is described by $\top$, its position should now be described by $\bot$, and vice-versa.

In this language, $\text{skip}$ represents that the program does nothing.

In this language, $s_1 ; s_2$ represents that the program first runs $s_1$ then it runs $s_2$.

2 **Prove Semantics Are Functional**

We wish for these semantics to describe exactly how a robot should move. This means that, for any given starting position, and any given program, your program describes exactly one ending position.

Put more formally, I wish you to prove that for all $\ell \in \mathbb{L}$ and for all $(x, b) \in \mathbb{R} \times \mathbb{B}$, there exists $(x', b') \in \mathbb{R} \times \mathbb{B}$ such that $\ell \vdash (x, b) \rightsquigarrow (x', b')$. Furthermore, show that for all $(x', b')$ and $(x'', b'') \in \mathbb{R} \times \mathbb{B}$, if $\ell \vdash (x, b) \rightsquigarrow (x', b')$ and $\ell \vdash (x, b) \rightsquigarrow (x'', b'')$ then $x' = x''$ and $b' = b''$.

3 **Write a Derivation Showing That the Semantics Operate Correctly**

Write a full derivation using your rules that shows:

$$\text{flip; (move 3; (skip; flip))) \vdash (3, \top) \rightsquigarrow (0, \top)}$$
4 Prove Anything Holds on Uninhabited Types

Consider the language $M$ described by the following syntax:

\[ s ::= \text{Continue}\ s \]

Prove via structural induction the theorem $\forall x \in M. \phi$ for any arbitrary $\phi$. (Hint 1: prove a stronger theorem first) (Hint 2: $\bot$ implies $\phi$ for all $\phi$).

5 Prove One Inductive Proposition Implies Another

Consider the language $N$ described by the following syntax:

\[ n ::= S\ n \mid O \]

Consider the inductive proposition $even\ n \subseteq N$ defined below:

\[
\begin{align*}
\text{even}\ O & \quad \text{EVENO} \\
\text{even}\ S\ n & \quad \text{EVENSS}
\end{align*}
\]

and consider the inductive proposition $\text{mod4}\ n \subseteq N$ defined below:

\[
\begin{align*}
\text{mod4}\ O & \quad \text{Mod4O} \\
\text{mod4}\ S\ S\ S\ S\ n & \quad \text{Mod4SSSS}
\end{align*}
\]

Prove that for all $n \in N$, $\text{mod4}\ n$ implies $\text{even}\ n$.

6 Prove One Inductive Proposition Implies Another Exists

Consider the language $T$ described by the following syntax:

\[ t ::= \text{Node}\ (t, t) \mid \text{Leaf} \]

Consider the inductive proposition $\text{right\_lean}\ t \subseteq T$ defined below:

\[
\begin{align*}
\text{right\_lean}\ \text{Leaf} & \quad \text{RIGHTLEANLEAF} \\
\text{right\_lean}\ \text{Node}(\text{Leaf}, t) & \quad \text{RIGHTLEANNODE}
\end{align*}
\]

Consider the inductive proposition $\text{line\_length}\ t = n \subseteq T \times N$ defined below:

\[
\begin{align*}
\text{line\_length}\ \text{Leaf} & = 0 \quad \text{LINELENGTHLEAF} \\
\text{line\_length}\ t = n & \quad \text{LINELENGTHLEAFF} \\
\text{line\_length}\ \text{Node}(t, \text{Leaf}) = n + 1 & \quad \text{LINELENGTHLEAFR}
\end{align*}
\]

Prove that for all $t \in T$, if $\text{right\_lean}\ t$ then there exists some $n \in N$ for which $\text{line\_length}\ t = n$ holds.