1 Progress Cases

Recall the statement of Progress: \( \forall e. \forall \tau. \text{if } \vdash e : \tau \text{ then } e \text{ value or there exists } e' \text{ such that } e \rightarrow e' \). Progress is proved via induction over the derivation of \( \vdash e : \tau \). Prove the following cases of Progress (you are welcome to use the IH on sub-derivations).

1. The last step of the derivation was:

\[
\frac{x : \tau_1 \vdash e : \tau_2}{\vdash \lambda(x : \tau).e : \tau_1 \rightarrow \tau_2}
\]

2. The last step of the derivation was:

\[
\frac{\vdash e_0 : \text{Bool} \quad \vdash e_1 : \tau \quad \vdash e_2 : \tau}{\vdash \text{if } e_0 \text{ then } e_1 \text{ else } e_2 : \tau}
\]

3. The last step of the derivation was:

\[
\frac{\vdash e : \text{Nat}}{\vdash \text{IsZero } e : \text{Bool}}
\]

2 The Substitution Lemma Cases

Recall the statement of The Substitution Lemma: \( \forall \Gamma. \forall e. \forall \tau. \forall \tau'. \forall v. \text{ if } \Gamma, x : \tau' \vdash e : \tau \text{ and } \Gamma \vdash v : \tau \text{ and } e[v/x] = e' \text{, then } \Gamma \vdash e' : \tau \).

The Substitution Lemma is proved via induction over the derivation of \( \Gamma, x : \tau' \vdash e : \tau \). Prove the following cases of The Substitution Lemma (you are welcome to use the IH on sub-derivations).

1. The last step of the derivation was:

\[
\frac{\Gamma, x : \tau' \vdash e_2 : \tau_1 \rightarrow \tau_2 \quad \Gamma, x : \tau' \vdash e_1 : \tau_1}{\Gamma, x : \tau' \vdash e_2 e_1 : \tau_2}
\]

2. The last step of the derivation was:

\[
\frac{\Gamma, x : \tau' \vdash e_0 : \text{Bool} \quad \Gamma, x : \tau' \vdash e_1 : \tau \quad \Gamma, x : \tau' \vdash e_2 : \tau}{\Gamma, x : \tau' \vdash \text{if } e_0 \text{ then } e_1 \text{ else } e_2 : \tau}
\]

3. The last step of the derivation was:

\[
\frac{\Gamma, x : \tau' \vdash e : \text{Nat}}{\Gamma, x : \tau' \vdash \text{IsZero } e : \text{Bool}}
\]
3 Preservation Cases

Recall the statement of Preservation: \( \forall e. \forall e'. \forall \tau. \text{if } \vdash e : \tau \text{ and } e \to e' \text{ then } \vdash e' : \tau. \)

Preservation is proved via induction over the derivation of \( e \to e' \). Prove the following cases of Preservation (you are welcome to use the IH on sub-derivations).

1. The last step of the derivation was:
\[
\frac{e_1 \to e'_1}{e_1 \ e_2 \to e'_1 \ e_2}
\]

2. The last step of the derivation was:
\[
\frac{v_1 \text{ value } \ e_2 \to e'_2}{v_1 \ e_2 \to v_1e'_2}
\]

3. The last step of the derivation was:
\[
\text{IsZero O } \to \text{ True}
\]

4 What’s Preserved

Recall that Progress is: \( \forall e. \forall \tau. \text{if } \vdash e : \tau \text{ then e value or there exists } e' \text{ such that } e \to e'. \)

Recall that Preservation is: \( \forall e. \forall e'. \forall \tau. \text{if } \vdash e : \tau \text{ and } e \to e' \text{ then } \vdash e' : \tau. \)

Recall that Normalization is: \( \forall e. \forall \tau \text{ if } \vdash e : \tau \text{ then there exists some } v \text{ such that value } v \text{ and } e \to^* v. \)

Recall that Unique Typing is: \( \forall \Gamma. \forall e. \forall \tau. \forall \tau'. \text{if } \Gamma \vdash e : \tau \text{ and } \Gamma \vdash e : \tau' \text{, then } \tau = \tau'. \)

In the following questions, I will add rules to possibly the typing derivation, to possibly the value derivation, to possibly the evaluation derivation, and to possibly the semantics. For each question, you should answer 4 things: (1) is Progress true in the updated language, (2) is Preservation true in the updated language, (3) is Normalization true in the updated language, (4) is Unique Typing true in the updated language.

1. 
\[
\frac{e_2 \to e'_2}{e_1 \ e_2 \to e_1 e'_2}
\]

2. Expressions are enhanced to have the additional construct: \( e := \ldots | \text{loop}. \)
\[
\Gamma \vdash \text{loop} : \tau \quad \text{loop } \to \text{ loop}
\]

3. 
\[
S \ O \to \text{ True}
\]