1 Substitution Lemma for Types

Prove that if $\Gamma, X : * \vdash \tau$ type and $\Gamma \vdash \tau' \text{ type}$, then $\Gamma \vdash \tau[X/\tau'] \text{ type}$. Note that you will need to use induction over the derivation of $\Gamma, X : * \vdash \tau$. You will need to use the definitions of variable replacement in types.

2 Product Types in System $F$

Define $\tau_1 \times \tau_2 := \forall X. (\tau_1 \rightarrow \tau_2 \rightarrow X) \rightarrow X$. Define $(e_1, e_2)$ as $\lambda X. \lambda (g : \tau_1 \rightarrow \tau_2 \rightarrow X). g \ e_1 \ e_2$. Define $\text{fst } e$ as $e[\tau_1][\lambda (x : \tau_1). \lambda (y : \tau_2). x]$. Define $\text{snd } e$ as $e[\tau_2][\lambda (x : \tau_1). \lambda (y : \tau_2). y]$.

For the following proofs you don’t need to include the derivations of variable replacement.

Prove:

1. If $\vdash e_1 : \tau_1$ and $\vdash e_2 : \tau_2$ then $\vdash (e_1, e_2) : \tau_1 \times \tau_2$
2. If $\vdash e : \tau_1 \times \tau_2$ then $\vdash \text{fst } e : \tau_1$
3. If $e_1 \rightarrow^* v_1$ and $e_2 \rightarrow^* v_2$ and value $v_1$ and value $v_2$ and $\vdash e_1 : \tau_1$ and $\vdash e_2 : \tau_2$ then $\vdash \text{fst } (e_1, e_2) \rightarrow^* v_1$.

You don’t need to include the derivation for each individual step, just show each of the steps involved in the evaluation.

3 The Unit Type

Come up with a type $\tau$ and an expression $e$, such that $\tau$ and $e$ satisfy the rules of the unit type. Namely:

$\vdash e : \tau$

value $e$

4 The Empty Type

Come up with a type $\tau$ and an expression $\text{coerce}$, such that $\tau$ and $\text{coerce}$ satisfy the rules of the empty type. Namely:

$e : \tau$

$\text{coerce}(e, \tau') : \tau'$

Does this type have any values? Argue one way or another (this argument can be informal – in other words this argument needn’t be a proof).