Automated Theorem Proving in Propositional Logic

Why does adding in $\Rightarrow$ and $\subseteq$ derivations help typecheck?

- On programs without annotations, the relationship between expressions and types is tighter.
- With a tighter relationship, it is easier to deduce one from the other.

Can we do the same for theorem proving?

Yes!

With this tighter relationship, we can even derive proofs from types.
In the proof-theoretic context, this is known as intercalation.

Intercalation formalism:

\[ \Gamma \vdash \phi \downarrow \quad \text{\textit{-} \phi \text{ is deduced}} \quad \Gamma \vdash e \in \chi \]

\[ \Gamma \vdash e \uparrow \quad \text{\textit{-} \phi \text{ is extracted}} \quad \Gamma \vdash e \Rightarrow \chi \]

\[ \Gamma, \phi \vdash \phi \downarrow \quad \text{(Ax)} \]

\[ \Gamma \vdash \phi \downarrow \]

\[ \Gamma \vdash \top \downarrow \quad \text{(I)} \]

\[ \Gamma \vdash \phi \uparrow \]

\[ \Gamma \vdash \perp \downarrow \quad \text{\textit{+} \phi \uparrow} \]

\[ \Gamma \vdash \phi \uparrow \]


\[ \Gamma, \alpha \vdash \gamma \Gamma \quad \Gamma \vdash \phi \rightarrow \Phi \rightarrow \Sigma \]

\[ \Gamma \vdash \phi \quad \Gamma \vdash \gamma \rightarrow \rightarrow \Sigma \]

\[ \Gamma \vdash \phi \quad \Gamma \vdash \psi \rightarrow \rightarrow \Lambda I \]

\[ \Gamma \vdash \psi \quad \Gamma \vdash \phi \rightarrow \rightarrow \Lambda I \]

\[ \Gamma \vdash \phi \varepsilon \psi \theta \rightarrow \rightarrow \Lambda I \]

\[ \Gamma \vdash \psi \rightarrow \rightarrow \Lambda E_c \]

\[ \Gamma \vdash \phi \varepsilon \psi \theta \rightarrow \rightarrow \Lambda E \]

\[ \Gamma \vdash \phi \varepsilon \psi \theta \rightarrow \rightarrow \Lambda E_r \]

\[ \Gamma \vdash \phi \varepsilon \psi \theta \rightarrow \rightarrow \Lambda E \]

\[ \Gamma \vdash \phi \varepsilon \psi \theta \rightarrow \rightarrow \Lambda E \]

\[ \Gamma \vdash \phi \varepsilon \psi \theta \rightarrow \rightarrow \Lambda E \]

\[ \Gamma \vdash \phi \varepsilon \psi \theta \rightarrow \rightarrow \Lambda E \]

Thus, if 
then 

\[ \Gamma \vdash \phi \]

\[ \Gamma \vdash \phi \]