Ordering without forbidden patterns

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Abstract

Let $\mathcal{F}$ be a set of ordered patterns, i.e., graphs whose vertices are linearly ordered. An $\mathcal{F}$-free ordering of the vertices of a graph H is a linear ordering of $V(H)$ such that none of patterns in $\mathcal{F}$ appears as an induced ordered subgraph. We denote by $\textsc{Ord}(\mathcal{F})$ the decision problem asking whether an input graph admits an $\mathcal{F}$-free ordering; we also use $\textsc{Ord}(\mathcal{F})$ to denote the class of graphs that do admit an $\mathcal{F}$-free ordering. It was observed by Damaschke (and others) that many natural graph classes can be described as $\textsc{Ord}(\mathcal{F})$ for sets $\mathcal{F}$ of small patterns (with three or four vertices). This includes bipartite graphs, split graphs, interval graphs, proper interval graphs, cographs, comparability graphs, chordal graphs, strongly chordal graphs, and so on. Damaschke also noted that for many sets $\mathcal{F}$ consisting of patterns with at most three vertices, $\textsc{Ord}(\mathcal{F})$ is polynomial-time solvable by known algorithms or their simple modifications. We complete the picture by proving that *all* these problems can be solved in polynomial time. In fact, we provide a single master algorithm, i.e., we solve in polynomial time the problem $\textsc{Ord}_3$ in which the input is a set $\mathcal{F}$ of patterns with at most three vertices and a graph H, and the problem is to decide whether or not H admits an $\mathcal{F}$-free ordering of the vertices. Our algorithm certifies non-membership by a forbidden substructure, and thus provides a single forbidden structure characterization for all the graph classes described by some $\textsc{Ord}(\mathcal{F})$ with $\mathcal{F}$ consisting of patterns with at most three vertices. This includes bipartite graphs, split graphs, interval graphs, proper interval graphs, chordal graphs, and comparability graphs. Many of the problems $\textsc{Ord}(\mathcal{F})$ with $\mathcal{F}$ consisting of larger patterns have been shown to be NP-complete by Duffus, Ginn, and Rodl, and we add two simple examples.

We also discuss a bipartite version of the problem, $\textsc{BiOrd}(\mathcal{F})$, in which the input is a bipartite graph H with a fixed bipartition of the vertices, and we are given a set $\mathcal{F}$ of bipartite patterns. We give a unified polynomial-time algorithm for all problems $\textsc{BiOrd}(\mathcal{F})$ where $\mathcal{F}$ has at most four vertices, i.e., we solve the analogous problem $\textsc{BiOrd}_4$. This is also a certifying algorithm, and it yields a unified forbidden substructure characterization for all bipartite graph classes described by some $\textsc{BiOrd}(\mathcal{F})$ with $\mathcal{F}$ consisting of bipartite patterns with at most four vertices. This includes chordal bipartite graphs, co-circular-arc bipartite graphs, and bipartite permutation graphs. We also describe some examples of digraph ordering problems and algorithms.

We conjecture that for every set $\mathcal{F}$ of forbidden patterns, $\textsc{Ord}(\mathcal{F})$ is either polynomial or NP-complete.

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1 Problem definition and motivation

For every positive integer k we write $[k] = \{1, 2, \dots, k\}$, $E_k = \{\{i, j\} \mid i, j \in [k], i \neq j\}$, and $\mathcal{F}_k = 2^{E_k}$. Each element in $\mathcal{F}_k$ can be viewed as a labeled graph on vertex set $[k]$ and is called a *pattern* of *order* k, or simply a *k-pattern*. Given an input graph H and a linear ordering $<$ of its vertices, we say that a pattern $\mathcal{F} \in \mathcal{F}_k$ *occurs* in H (under the ordering $<$) if H contains vertices $v_1 < v_2 < \cdots < v_k$ such that the induced ordered subgraph on these vertices is isomorphic to F, i.e., for every $i, j \in [k]$, $v_i v_j \in E(H)$ if and only if $\{i, j\} \in F$. For convenience, we shall henceforth write ij to simplify notation for unordered pairs $\{i, j\}$.

We say that a linear ordering $<$ of $V(H)$ is s *$\mathcal{F}$-free* if none of the forbidden patterns in $\mathcal{F}$ appears in $<$. The problem $\textsc{Ord}(\mathcal{F})$ asks whether or not the input graph H has an $\mathcal{F}$-free ordering, and the problem $\textsc{Ord}_k$ asks, for an input $\mathcal{F} \in \mathcal{F}_k$ and a graph H, whether or not H has an $\mathcal{F}$-free ordering.

Ordering problems can be represented as a satisfiability problem by using variables W_{uv} ($u, v \in V(H)$, $u \neq v$), where we interpret W_{uv} as $u < v$ and then ask that $W_{vu} = \neg W_{uv}$ and that transitivity holds: $(W_{uv} \wedge W_{vw}) \Rightarrow W_{uw}$. Forbidden 3-patterns can be represented by using 2-clauses: whenever vertices x, y, z induce a forbidden pattern, we ask that $W_{yx} \vee W_{zy}$ is satisfied. In this sense, $\textsc{Ord}_3$ can always be formulated as a *2-SAT problem with ordering*, where we solve 2-satisfiability under the additional condition that the variables correspond to a linear order. Similarly, patterns of order k can be represented by $(k-1)$-clauses and we obtain more general satisfiability problems with ordering. Note that even in the case of $k = 3$ we need to include clauses of the form $(W_{uv} \wedge W_{vw}) \Rightarrow W_{uw}$ to accommodate the transitivity, so we are not dealing with instances of 2-SAT, and Horn Clauses.

The problems $\textsc{Ord}(\mathcal{F})$ have been studied by Damaschke [5], Duffus, Ginn, and Rodl [6], and others. In particular, Damaschke lists many graph classes which can be equivalently described as $\textsc{Ord}(\mathcal{F})$. For example [2], it is well known that a graph H is chordal if and only if it admits an $\mathcal{F}$-free ordering for $\mathcal{F}$ consisting of the single pattern $\{12, 13\}$, and H is an interval graph if and only if it admits an $\mathcal{F}$-free ordering for $\mathcal{F}$ consisting of the pattern $\{\{13\}, \{13, 23\}\}$.

Similar sets of patterns from $\mathcal{F}_3$ describe proper interval graphs, bipartite graphs, split graphs, and comparability graphs [5]. With patterns from $\mathcal{F}_4$ we can additionally describe strongly chordal graphs [7], circular-arc graphs [20], and many other graph classes.

Let $E'_k = \{\{i, j'\} \mid i \in [\ell], j' \in \{1', 2', \dots, \ell'\}\}$, $\ell + \ell' \leq k$ and $\mathcal{B}_k = 2^{E'_k}$. For convenience we denote pairs $\{i, j'\}$ by ij'. Each element in $\mathcal{B}_k$ is called a *bipartite pattern* and can be viewed as a bipartite graph whose vertices in each part of the bipartition are ordered. One can define ordering problems in bipartite graphs as follows. The input is a bipartite graph H with a fixed bipartition $V(H) = U \cup V$, and a set $\mathcal{F} \in \mathcal{B}_k$, and the question is whether or not the sets U and V can be ordered so that no pattern from $\mathcal{F}$ occurs. If $\mathcal{F}$ is fixed, we have the problem $\textsc{BiOrd}(\mathcal{F})$; we also define $BORD_k$ when $\mathcal{F} \in B_k$ is part of the input. Several known bipartite graphs classes can be characterized as $\textsc{BiOrd}(\mathcal{F})$ for $\mathcal{F} \in \mathcal{B}_4$. For instance, $\mathcal{F} = \{11', 31'\}$ yields $\textsc{BiOrd}(\mathcal{F})$ to consist of precisely convex bipartite graphs, and $\mathcal{F} = \{\{11', 12', 21'\}, \{12', 21'\}, \{12', 21', 22'\}\}$ similarly defines bipartite permutation graphs (i.e., proper interval bigraphs) [14, 23, 24]. One can similarly obtain the classes of chordal bipartite graphs, and bipartite co-circular arc bigraphs [16].

Summary of our main results

We show that $\textsc{Ord}_3$ and $\textsc{BiOrd}_4$ are solvable in polynomial time. In particular, this completes the picture analyzed by Damaschke [5], and proves that *all* $\textsc{Ord}(\mathcal{F})$ with $\mathcal{F} \in \mathcal{F}_3$ are polynomial-time solvable; similarly, all $\textsc{BiOrd}(\mathcal{F})$ with $\mathcal{F} \in \mathcal{B}_4$ are polynomial-time solvable.

Both of these settings give rise to a general class of 2-SAT problems where we impose an additional condition that the truth assignment gives a partial order on a set V when the variables correspond to pairs of elements of V.

We also discuss digraphs with forbidden patterns on three vertices, amd present two classes of digraphs for which our algorithm can be deployed to obtain the desired ordering without forbidden patterns.

We further illustrate sets $\mathcal{F} \in \mathcal{F}_4$ for which $\textsc{Ord}(\mathcal{F})$ is polynomial time solvable and other sets $\mathcal{F} \in \mathcal{F}_4$ for which the same problem is NP-complete. Many more NP-complete cases of $\textsc{Ord}(\mathcal{F})$ are presented in [6]; in particular, the authors of [6] conjecture that any $\mathcal{F}$ consisting of a single 2-connected pattern (other than a complete graph) yields an NP-complete $\textsc{Ord}(\mathcal{F})$.

Our master algorithm for $\textsc{Ord}_3$ provides a unified approach to all recognition problems for classes $\textsc{Ord}(\mathcal{F})$ with $\mathcal{F} \in \mathcal{F}_3$, including the classes of split graphs, chordal graphs, interval graphs, proper interval graphs, and comparability graphs. Our algorithm is a certifying algorithm, and so it also provides a unified obstruction characterization for all these graph classes. A similar situation occurs with $\textsc{BiOrd}(\mathcal{F})$ with $\mathcal{F} \in \mathcal{B}_4$ and classes characterized as $\textsc{BiOrd}(\mathcal{F})$ with $\mathcal{F} \in \mathcal{B}_4$, including the classes of convex bipartite graphs, bipartite permutation graphs, chordal bipartite graphs, and bipartite co-circular arc bigraphs. These graph classes received much attention in the past; efficient recognition algorithms and structural characterizations can be found in [1, 3, 4, 10, 15, 21, 23, 25] and elsewhere, cf. [2, 13].

The algorithm for $\textsc{Ord}_3$ runs in time $O(n^3)$ where n is the number of vertices of H and in several cases (depending on $\mathcal{F}$) it runs in time $O(nm)$, where m is the number of edges of H. The algorithm for $\textsc{BiOrd}_4$ runs in time $O(n^4)$ and in several cases in time $O(n^2m)$. The algorithms employ an auxiliary digraph, similar to auxiliary digraphs in [9, 17].

We conjecture that for every set $\mathcal{F}$ of forbidden patterns, $\textsc{Ord}(\mathcal{F})$ is either polynomial or NP-complete and provide some additional evidence for this dichotomy.

2 Algorithm for $\textsc{Ord}_3$ on undirected graphs

Consider an input graph H and a forbidden pattern $\mathcal{F} \in \mathcal{F}_3$. $\mathcal{F}$ imposes a constraint on every three vertices x, y, z of H. This means that whenever (x, y, z) induce the subgraph given in $\mathcal{F}$ and x is before y then z must be before y.

We first construct an auxiliary digraph H^+, so called *constraint digraph*. The vertex set of H^+ consists of the pairs $(x, y) \in V(H) \times V(H)$, $x \neq y$, and the arcs of H^+ are defined as follows. There is an arc from (x, y) to (z, y) and an arc from (y, z) to (y, x) whenever the vertices x, y, z ordered as $x < y < z$ induce a forbidden pattern in $\mathcal{F}$. We say that a pair (x, y) *dominates* (x', y') and we write $(x, y) \to (x', y')$ if there is an arc from (x, y) to (x', y') in H^+.

Consider a strong component S of H^+. The dual component $\overline{S}$ of S consists of all the pairs (y, x) where $(x, y) \in S$. Note that if $(x, y) \to (x, z)$, then $(z, x) \to (y, x)$.

There are two operations that appear naturally when dealing with orderings and forbidden patterns [5]. If we replace each pattern in $\mathcal{F}$ with its *complement* (change edges to nonedges and vice versa), thus obtaining a set $\overline{\mathcal{F}}$, then a linear ordering of $V(H)$ is $\mathcal{F}$-free for H if and only if it is $\overline{\mathcal{F}}$-free for the complementary graph $\overline{H}$. Another equivalence is obtained by replacing $\mathcal{F}$ with patterns that represent the same induced subgraphs but with the reversed order, e.g., replace $\{12, 13\}$ by $\{32, 31\}$. Then a linear ordering will be $\mathcal{F}$-free if and only if the reverse ordering will be free for the reverse patterns. We will rely on these two properties in some of our proofs.

In general, the structure of the digraph H^+ depends on the patterns. It is easy to see that if $\{12, 23\}$ or $\{13\}$ is the only forbidden pattern in $\mathcal{F} \subset \mathcal{F}_3$, then $(u, v)(u', v')$ is an arc of H^+ if and only if $(u', v')(u, v)$ is an arc of H^+, i.e. $(u, v)(u', v')$ is a symmetric arc of H^+ and hence H^+ is a graph. On the other hand, if $\{12, 13\}$ is the only forbidden pattern in $\mathcal{F}$, H^+ is a digraph without digons and if $(u, v)(u', v')$ is an arc, then $(u', v')(u, v)$ is not an arc of H^+.

To keep up with linear orderings when interpreting vertices of $V(H^+)$ as ordering constraints, we introduce the following definition. If all pairs $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_0)$, $n \geq 1$, are in the same subset D of $V(H^+)$ then we say that $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_0)$ is a *circuit* in D.

Lemma 2.1 *Let $\mathcal{F} \subseteq \mathcal{F}_3$ and let H^+ be the constraint digraph of H with respect to $\mathcal{F}$. If there exists a circuit in a strong component S of H^+, then H has no $\mathcal{F}$-free ordering.*

Proof: For a contradiction suppose $<$ is an $\mathcal{F}$-free ordering. Consider a circuit $(x_0, x_1), \ldots,$ $(x_{n-1}, x_n), (x_n, x_0)$ in S. Since S is strong, there is a directed path P_i from (x_i, x_{i+1}) to (x_{i+1}, x_{i+2}) in S. If $x_i < x_{i+1}$ then following the path P_i in S we conclude that we must have $x_{i+1} < x_{i+2}$, and eventually by following each P_j, $0 \leq j \leq n$ we conclude that $x_i < x_{i+1} < \cdots < x_{i-1} < x_i$. This is a contradiction. Thus we must have $x_{i+1} < x_i$. Now there is a path P'_i in the $\overline{S}$ and hence by following the path P'_i we must have $x_i < x_{i-1}$ and eventually conclude that $x_{i+1} < x_i < \cdots < x_{i+2} < x_{i+1}$, yielding a contradiction. $\diamond$

Lemma 2.2 (a) *Suppose $\emptyset \in \mathcal{F} \in \mathcal{F}_3$.*

If H contains an independent set of three vertices, then H^+ has a strong component with a circuit and H has no $\mathcal{F}$-free ordering.

Otherwise H^+ is the same for $\mathcal{F}$ and for $(\mathcal{F} \setminus \{\emptyset\})$, and H has an $\mathcal{F}$-free ordering if and only if it has an $(\mathcal{F} \setminus \{\emptyset\})$-free ordering.

(b) *Suppose $\{12, 13, 23\} \in \mathcal{F} \in \mathcal{F}_3$.*

If H contains a triangle, then H^+ has a strong component with a circuit and H has no $\mathcal{F}$-free ordering.

Otherwise H^+ is the same for $\mathcal{F}$ and for $(\mathcal{F} \setminus \{\emptyset\})$, and H has an $\mathcal{F}$-free ordering if and only if it has an $(\mathcal{F} \setminus \{\{12, 13, 23\}\})$-free ordering.

Proof: We only prove part (a) since the proof of (b) is similar.

Let a, b, c be pairwise nonadjacent vertices of H. If $\emptyset \in \mathcal{F}$, then $(a,b) \to (c,b)$, and $(c,b) \to (a,b)$, thus (a,b) and (c,b) are in the same strong component of H^+. Similarly, we have $(a,b) \to (a,c)$, and $(a,c) \to (a,b)$, thus (a,b) and (a,c) are in the same strong component of H^+. By symmetry, applied to other pairs, we conclude that all ordered pairs of two distinct vertices from the set $\{a,b,c\}$ are in the same strong component S of H^+. Clearly, $(a,b),(b,a)$ is a circuit in S.

As for the second part of the claim, if H has no independent set of three vertices, then $\emptyset$ contributes no restriction to orderings of $V(H)$, so both the claims follow. $\diamond$

Our main result is the following theorem which implies that $\textsc{Ord}_3$ is solvable in polynomial time. (In fact, its proof amounts to a polynomial-time algorithm.)

Theorem 2.3 *Let $\mathcal{F} \in \mathcal{F}_3$ and let H^+ be the constraint digraph of H with respect to $\mathcal{F}$. Then H has an $\mathcal{F}$-free ordering if and only if no strong component of H^+ contains a circuit.*

Theorem 2.3 will follow from the correctness of our algorithm for $\textsc{Ord}_3$. The algorithm and its proof of correctness comprise the rest of this section. Note that Theorem 2.3 provides a universal forbidden substructure (namely a circuit in a strong component of H^+) characterizing the membership in graph classes as varied as chordal graphs, interval graphs, proper interval graphs, comparability graphs, and co-comparability graphs.

We say a strong component S of H^+ is a *sink component* if there is no arc from S to a vertex outside S in H^+. Consider a subset D of the pairs in $V(H^+)$. We say that a strong component S of $H^+ \setminus (D \cup \overline{D})$ is *green with respect to* D if there is no arc from an element of S to a vertex in $H^+ \setminus (D \cup \overline{D} \cup S)$. This is equivalent to the condition that S is a sink component in $H^+ \setminus (D \cup \overline{D})$.

In the algorithm below, we start with an empty set D and we construct the final set D step by step. After each step of the algorithm, D (and hence also $\overline{D}$) is the union of vertex-sets of strong components of H^+ and neither D nor $\overline{D}$ contains a circuit. Each strong component S of H^+ either belongs to D or $\overline{D}$ or $V(H^+) \setminus (D \cup \overline{D})$. At the end of the algorithm $D \cup \overline{D}$ is a partition of the vertices (pairs) in $V(H^+)$ such that whenever $(x,y),(y,z) \in D$ then $(x,z) \in D$. We will say that D satisfies *transitivity condition.* At the end of the algorithm we place x before y whenever $(x,y) \in D$ and we obtain the desired ordering. We say a strong component is *trivial* if it has only one element otherwise it is called *non-trivial.*

Ordering with forbidden 3-patterns, $\textsc{Ord}_3$
Input: A graph H and a set $\mathcal{F} \subseteq \mathcal{F}_3$ of forbidden patterns on three vertices
Output: An $\mathcal{F}$-free ordering of the vertices of H or report that there is no such ordering.

Algorithm for $\textsc{Ord}_3$

1. If a strong component S of H^+ contains a circuit then report that no solution exists and exit. Otherwise, remove $\emptyset$ and $\{12, 13, 23\}$ from $\mathcal{F}$. If $\mathcal{F}$ is empty after this step, then return any ordering of vertices of H and stop.

2. Set D to be the empty set.

3. Choose a strong component S of H^+ that is green with respect to D. The choice is made according to the following rules.

a) If $\mathcal{F}$ contains one of the forbidden patterns $\{13, 23\}, \{12, 13\}, \{12, 23\}$, then the priority is given to strong components containing a pair (x, y) with $xy \in E(H)$. If there is a choice then it is preferred S to be a trivial component. Subject to these preferences, if there are several candidates, then priority is given to the ones that are sink components in H^+.

b) If $\mathcal{F}$ contains one of $\{12\}, \{23\}, \{13\}$, then priority is given to a strong component S containing (x, y) with $xy \notin E(H)$. If there is a choice, then the priority is given to trivial components, and if there are several candidates for S, then preference is given to the sink components in H^+.

4. If by adding S into D we do not close a circuit, then we add S into D and discard $\overline{S}$. Otherwise we add $\overline{S}$ and its outsection (all vertices in H^+ that are reachable from $\overline{S}$) into D and discard S and its insection (the vertices that can reach S). Return to Step 3 if there are some strong components of H^+ left.

5. For every $(x, y) \in D$, place x before y in the final ordering.

We now prove correctness of the algorithm which, in particular, implies Theorem 2.3. The validity of the first step of the algorithm is justified by Lemmas 2.1 and 2.2. Thus we may assume from now on that no strong component of H^+ contains a circuit, that $\emptyset$ and $\{12, 13, 23\}$ are not in $\mathcal{F}$ and that $\mathcal{F} \neq \emptyset$.

Observe that a strong component S of H^+ contains a circuit if and only if $\overline{S}$ contains a circuit. Moreover, $S \cap \overline{S} = \emptyset$ as otherwise for every $(x_0, x_1) \in S \cap \overline{S}$, we have $(x_1, x_0) \in S \cap \overline{S}$ and hence there would be a circuit $(x_0, x_1), (x_1, x_0)$ in S.

We first need the following lemma about the structure of strong components of H^+.

Lemma 2.4 *If $\mathcal{F}$ has only one element and that element is one of $\{12, 23\}, \{13\}$, then H^+ is symmetric (it is just a graph).*

Proof: We prove then lemma when $\mathcal{F} = \{\{12, 23\}\}$ and the proof for the other case is obtained by applying the arguments in the complement of H. Suppose $(x, y) \to (z, y)$ according to $\{12, 23\}$. Thus by definition of H^+ we have $xy, yz \in E(H)$ and $xz \notin E(H)$. By considering the order $z < y < x$, we conclude that $(z, y) \to (x, y)$. Hence, $(x, y)(z, y)$ is a symmetric arc. Now suppose $(x, y) \to (x, z)$ according to $\{12, 23\}$. Thus by definition of H^+ we have $xz, xy \in E(H)$ and $yz \notin E(H)$ and hence $(x, z) \to (x, y)$ implying that $(x, y)(x, z)$ is a symmetric arc. ⋄

Claim 2.5 *Let $C = w_1 w_2 \dots w_k w_1$ be an induced cycle of length $k \geq 4$ in H and let $\{13, 23\} \in \mathcal{F}$. Then the strong component of H^+ containing (w_1, w_2) contains a circuit.*

Proof: To prove this claim, we first observe that $(w_i, w_{i+1}) \to (w_i, w_{i+2}) \to (w_{i+1}, w_{i+2})$ for $0 \leq i \leq k - 1$ (where all indices are taken modulo k). Hence, (w_1, w_2), (w_2, w_3), $\dots$, (w_{k-1}, w_k), (w_k, w_1) are all in the same strong component of H^+, and thus there exists a circuit in this strong component. This proves the claim. ⋄

We will use the following notation. For $x \in V(H)$, we let $N(x)$ be the set of all neighbors of x in H.

Claim 2.6 *If (x, y) does not dominate any pair in H^+, then one of the following happens:*

1) $N(x) \setminus \{y\} \subseteq N(y) \setminus \{x\}$,

2) $N(y) \setminus \{x\} \subseteq N(x) \setminus \{y\}$, *or*

3) (y, x) *does not dominate any pair in* H^+.

Proof: We show the argument when xy is an edge of H and the case $xy \notin E(H)$ follows by applying the argument in the complement of H. Suppose none of conditions 1) and 2) happens. Then there exists $z \in N(x) \setminus (N(y) \cup \{y\})$ and there exists $w \in N(y) \setminus (N(x) \cup \{x\})$. Now $\{12, 13\}$ is not in $\mathcal{F}$ as otherwise (x, y) would dominate the pair (z, y) in H^+. Similarly, none of $\{12, 23\}$ and $\{13, 23\}$ is a forbidden pattern in $\mathcal{F}$ as otherwise (x, y) would dominate the pair (x, z) or the pair (x, w) (respectively). Thus, we may assume that $\mathcal{F} \subseteq \{\{12\}, \{23\}, \{13\}\}$. If $\{12\} \in \mathcal{F}$, then there is no vertex $v \in V(H)$ outside $N(x) \cup N(y)$ as otherwise $(x, y) \to (v, y)$, a contradiction. Thus (y, x) does not dominate any pair in H^+ since for every $v \neq x, y$, the subgraph induced on v, x, y contains at least two edges. Hence 3) holds. Similar argument works if $\{23\} \in \mathcal{F}$. Finally, if $\mathcal{F} = \{\{13\}\}$, then by the assumption and by Lemma 2.4, none of (x, y) and (y, x) dominates a pair in H^+ and hence 3) holds. $\diamond$

Lemma 2.7 *The Algorithm for* ORD$_3$ *does not create a circuit in* D.

Proof: Suppose that by adding a green component S into D we close a circuit $C : (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ in $D \cup S$ for the first time. We may assume that n is minimum and $(x_n, x_0) \in S$. We also assume that $\mathcal{F}$ contains one of the patterns $\{13, 23\}, \{12, 13\}, \{12, 23\}$. This follows from the fact that we may apply our arguments for the complementary forbidden patterns in the complement of H. This choice will enable us to concentrate on Case (a) in Step 3 of the algorithm.

The proof is divided into the following four cases:

(A) S is a trivial component and $x_n x_0$ is an edge.

(B) S is a trivial component and $x_n x_0$ is not an edge.

(C1) S is a non-trivial component and $\{13, 23\} \in \mathcal{F}$ (or, by symmetry, $\{12, 13\} \in \mathcal{F}$).

(C2) S is a non-trivial component and $\{12, 23\} \in \mathcal{F}$.

The proof will show that when the first circuit $C : (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ is created, then the shortest circuit created at this time has $x_0, x_1, \ldots, x_n$ either induce a clique or induce an independent set with special adjacencies to the other vertices of H. This will imply that $\overline{S}$ and its outsection can be added into D without creating a circuit.

A. Suppose that (x_n, x_0) forms a trivial strong component S in H^+, and $x_n x_0$ is an edge of H.

First suppose (x_n, x_0) does not dominate any pair in H^+. Now according to the algorithm each pair (x_i, x_{i+1}), $0 \leq i \leq n$ is also in a trivial strong component and it does not dominate any other pair in H^+ and $x_i x_{i+1}$ is an edge of H. We show that (x_0, x_n) is also in a trivial component and it is green. For a contradiction suppose (x_0, x_n) dominates a pair in H^+. Since (x_n, x_0) does not dominate any pair in H^+ and by assumption (x_0, x_n) dominates a pair in H^+, Claim 2.6 implies that either $N(x_n) \setminus \{x_0\} \subset N(x_0) \setminus \{x_n\}$ or $N(x_0) \setminus \{x_n\} \subset N(x_n) \setminus \{x_0\}$. W.l.o.g assume $N(x_n) \setminus \{x_0\} \subset N(x_0) \setminus \{x_n\}$. Now $N(x_0) \setminus \{x_n\} \not\subseteq N(x_n) \setminus \{x_0\}$ as otherwise (x_0, x_n)

does not dominate any pair in H^+. Now these together with Lemma 2.4 imply that $\{12, 13\} \in \mathcal{F}$. Since (x_0, x_1) does not dominate any pair in H^+, we conclude that $N(x_0) \setminus \{x_1\} \subseteq N(x_1) \setminus \{x_0\}$ and by continuing this argument we conclude that $N(x_{n-1}) \setminus \{x_n\} \subseteq N(x_n) \setminus \{x_{n-1}\}$. Therefore $N(x_0) \setminus \{x_n\} \subseteq N(x_n) \setminus \{x_0\}$, a contradiction.

Therefore $S = \{(x_0, x_n)\}$ is a green strong component. If by adding (x_0, x_n) into D we close a circuit $C_1 : (x_n, y_1), (y_1, y_2), \ldots, (y_m, x_0)$ then there would be an earlier circuit

$$(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, y_1), (y_1, y_2), \ldots, (y_m, x_0),$$

a contradiction (Note that since $(x_0, x_n), (x_n, x_0)$ are singleton, all the pairs in C, C_1 apart from $(x_0, x_n), (x_n, x_0)$ are already in D).

Now we continue by assuming (x_n, x_0) is a singleton component and $x_n x_0$ is an edge and (x_n, x_0) dominates a pair in H^+.

According to the Algorithm, (x_i, x_{i+1}) is also in a trivial component and $x_i x_{i+1}$, $0 \leq i \leq n$ is an edge. Moreover, (x_0, x_n) is in a trivial component and it dominates a pair in H^+ as otherwise it should have been considered before (x_n, x_0).

Claim 2.8 *$x_0, x_1, \ldots, x_n$ induce a clique in H.*

Proof: Consider the edge $x_i x_j$, $i \neq j - 1$ in H (note that such an edge exists since $x_i x_{i+1}$, $0 \leq i \leq n$ is an edge). Whenever $\{12, 23\}$ is a forbidden pattern in $\mathcal{F}$, $x_{i-1} x_j$ is an edge of H as otherwise (x_{i-1}, x_i) dominates (x_j, x_i) and hence $(x_j, x_i) \in D \cup S$, implying a shorter circuit $(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), \ldots, (x_{j-1}, x_j), (x_j, x_i)$ in $D \cup S$ (the indices are modul n). Whenever $\{13, 23\}$ is a forbidden pattern, $x_{i-1} x_j$ is an edge of H as otherwise (x_{i-1}, x_i) dominates (x_{i-1}, x_j) and $(x_{i-1}, x_j) \to (x_i, x_j)$ and hence both $(x_{i-1}, x_j), (x_i, x_j) \in D \cup S$, implying a shorter circuit in $D \cup S$. By applying this argument when one of the $\{12, 23\}, \{13, 23\}$ is in $\mathcal{F}$ we conclude that $x_r x_j$ is an edge for every $r \neq j$. Therefore $x_0, x_1, \ldots, x_n$ induce a clique in H. Now suppose $x_i x_j$ is an edge for $j \neq i + 1$ (note that such an edge exists since $x_i x_{i+1}$, $0 \leq i \leq n$ is an edge). Whenever $\{12, 13\}$ is a forbidden pattern, $x_{i+1} x_j$ is an edge as otherwise (x_i, x_{i+1}) dominates (x_j, x_{i+1}) and hence we obtain a shorter circuit in $D \cup S$. Since $x_i x_{i+1}, x_{i+1} x_{i+2}$ are edges of H, by similar argument we conclude that $x_r x_j$ is an edge for every $r \neq j$. Therefore $x_0, x_1, \ldots, x_n$ induce a clique in H. $\diamond$

Note that since (x_n, x_0) is in a trivial component and (x_n, x_0) dominates a pair in H^+, by Lemma 2.4 either $\{13, 23\}$ or $\{12, 13\}$ is in $\mathcal{F}$ and $\{12, 23\} \notin \mathcal{F}$. Therefore we consider the case $\{13, 23\} \in \mathcal{F}$ and the argument for $\{12, 13\} \in \mathcal{F}$ is followed by symmetry.

$\{13, 23\}$ is a forbidden pattern in $\mathcal{F}$.

First suppose (x_0, x_n) dominates a pair according to $\{13, 23\}$. Thus there exists p_n such that $x_n p_n$ is an edges of H and $x_0 p_n$ is not an edge of H. Therefore there exists a smallest index $1 \leq i \leq n$ such that $p_n x_i$ is an edge and $p_n x_{i-1}$ is not an edge. Now $(x_{i-1}, x_i) \to (x_i, p_n)$ and $(x_i, p_n) \to (x_n, p_n)$. This would imply that $(p_n, x_n) \in D$.

If $i = 1$ then both $(x_0, p_n), (x_n, p_n)$ are in D and hence (x_0, x_n) does not dominates a pair outside D according to $\{13, 23\}$. Therefore we may assume $i > 1$ and (x_0, p_n) dominates some pair (w, p_n). We must have $w x_i \in E(H)$ as otherwise $(x_0, w), (w, p_n), (p_n, x_i), (x_i, x_0)$ is a circuit in a strong component of H^+. Now when $x_{i-1} w$ is not an edge of H, $(x_{i-1}, x_i) \to (x_{i-1}, w)$ and hence $(x_{i-1}, w) \to (x_0, w) \to (x_0, p_n) \in D$ implying (x_0, x_n) is also green. Thus we may assume that wx_{i-1} is an edge. Now $(x_{i-1}, p_n) \to (w, p_n)$, implying that $(w, p_n) \in D$. These imply that the only pair that (x_0, x_n) may dominate outside D is (x_0, p_n). However if there is no other forbidden pattern in $\mathcal{F}$, by adding (x_0, p_n) into D we do not close a circuit. Otherwise such a circuit comprises of the pairs $(p_n, y_1), (y_1, y_2), \ldots, (y_m, x_0)$ in D. Now

$$(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, p_n), p_n, y_1), (y_1, y_2), \ldots, (y_m, x_0)$$

yield an earlier circuit in D, a contradiction. Thus by adding (x_0, p_n) into D and then adding (x_0, x_n) into D we do not close a circuit into D as otherwise we conclude that there would be an earlier circuit in D.

Now we continue by assuming there is another forbidden pattern in $\mathcal{F}$. We show that in this case there is no pair dominated by (x_0, x_n) outside D according to this forbidden pattern. First suppose $\{12\}$ is also a forbidden pattern in $\mathcal{F}$ and (x_0, x_n) dominates some pair (q_n, x_n) according to $\{12\}$, where q_n is not adjacent to any of x_0, x_n. We note that (x_n, x_0) dominates (q_n, x_0) and since (x_n, x_0) is singleton, $(q_n, x_0) \in D$. We show that $(q_n, x_n) \in D$. Observe that we may assume that $x_{n-1}q_n \in E(H)$ as otherwise $(x_{n-1}, x_n) \to (q_n, x_n)$ and hence $(q_n, x_n) \in D$ and we are done. Now $x_{q-2}q_n$ must be an edge of H as otherwise (x_{n-2}, x_{n-1}) dominates (x_{n-2}, q_n) and we obtain an earlier circuit $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-3}, x_{n-2}), (x_{n-2}, q_n), (q_n, x_0)$ in D. By continuing this argument we conclude that $q_n x_0$ must be an edge as otherwise we obtain an earlier circuit in D, a contradiction. Therefore (x_0, x_n) is also green and hence by adding (x_0, x_n) into D we do not close a circuit as otherwise there would be an earlier circuit in D.

Analogously we conclude that if $\{23\}$ is a forbidden pattern then we can add (x_0, x_n) into D without creating a circuit. Note that (x_0, x_n) does not dominate any pair using the forbidden pattern $\{13\}$.

The last remaining case is when $\{12, 13\}$ is also a forbidden pattern. In this case (x_0, x_n) does not dominate any (z, x_n) in $H^+ \setminus (D \cup \overline{D})$ according to $\{12, 13\}$ $(zx_0 \in E(H),\ zx_n \notin E(H))$ as otherwise (x_n, x_0) dominates (x_n, z), and hence $(z, x_n) \in \overline{D}$, a contradiction.

B. Suppose that (x_n, x_0) forms a trivial strong component S in H^+, and $x_n x_0$ is not an edge of H.

According to the algorithm if $x_n x_0$ is not an edge then (x_0, x_n) does not dominate any pair in H^+ according to $\{12, 13\}, \{13, 23\}$. Moreover if there exists q such that $qx_0 \in E(H)$ and $qx_n \notin E(H)$ then according to the algorithm none of $\{12\}, \{23\}$ is in $\mathcal{F}$. Otherwise since qx_0 is an edge then one of (q, x_0) or (x_0, q) should have been considered earlier and hence one of them is in D. This would imply that (x_n, x_0) or (x_0, x_n) is also dominated by one (q, x_0), (x_0, q) and hence one of the $(x_0, x_n), (x_n, x_0)$ is in D a contradiction. Similarly if there exist q' such that $q'x_n$ is an edge and $q'x_0$ is not an edge we conclude that none of $\{12\}, \{23\}$ is in $\mathcal{F}$. Thus either (x_0, x_n) is green and in this case by adding (x_0, x_n) into D we don't create a circuit as otherwise there would be an earlier circuit in D or $\{13\} \in \mathcal{F}$ which implies that (x_n, x_0) is in a non-trivial component, a contradiction.

C. Suppose that (x_n, x_0) belongs to a non-trivial strong component S in H^+.

Case 1. $\{13, 23\} \in \mathcal{F}$.

First suppose $x_0 x_n$ is an edge of H. Since (x_n, x_0) is in a non-trivial component, there must be some other forbidden pattern in $\mathcal{F}$. We first consider the case that one of the $\{12\}, \{23\}, \{13\}$

is also in $\mathcal{F}$. As a consequence $x_0, x_1, \ldots, x_n$ induce a clique (To see this: x_1 must be adjacent to both x_0, x_n otherwise $(x_n, x_0) \to (x_1, x_0)$ for $\{12\} \in \mathcal{F}$ or $(x_0, x_1) \to (x_0, x_n)$ for $\{23\} \in \mathcal{F}$ or $(x_0, x_1) \to (x_n, x_1)$ for $\{13\} \in \mathcal{F}$ and hence in any case we obtain a shorter circuit. Now it is easy to see that (using the argument in Claim 2.8) both x_0x_1, x_nx_1 must be edges of H and by continuing this argument $x_0, x_1, \ldots, x_n$ induce a clique in H).

Since (x_n, x_0) is in a non-trivial component, there exists some pair $(u, v) \in S$ that dominates (x_n, x_0). First suppose $(u, v) \to (x_n, x_0)$ according to $\{13, 23\}$. In this case there exists q_n such that $q_nx_n \in E(H)$ and $x_0q_n \notin E(H)$ and $(q_n, x_0) \to (x_n, x_0)$. Now $q_nx_j \notin E(H)$, $j \neq 0, n$ as otherwise $(q_n, x_0) \to (x_j, x_0)$ and hence we have a shorter circuit $(x_0, x_1), (x_1, x_2), \ldots, (x_{j-1}, x_j), (x_j, x_0)$ in $D \cup S$. Observe that $(x_{n-1}, x_n) \to (x_{n-1}, q_n) \to (x_n, q_n)$. Now if $\{13\} \in \mathcal{F}$ then $(q_n, x_0) \to (q_n, x_{n-1})$ and hence we have a shorter circuit in $D \cup S$. If $\{12\} \in \mathcal{F})$ then $(x_{n-2}, x_{n-1}) \to (q_n, x_{n-1})$, and again there would be a shorter circuit in $D \cup S$. Finally when $\{23\} \in \mathcal{F}$ we have $(x_0, x_1) \to (x_0, q_n)$ while $(q_n, x_0) \in D \cup S$, a contradiction. Therefore we may assume that $(x, y) \in S$ dominates (x_n, x_0) according to one of the $\{12\}, \{23\}, \{13\}$. First suppose (x_n, q_n) dominates (x_n, x_0) according to $\{12\}$. Note that $q_nx_0, q_nx_n \notin E(H)$. Now $(x_n, x_0) \to (q_n, x_0)$. We show that q_n is not adjacent to any of $x_0, x_1, \ldots, x_n$. For a contradiction suppose there exists x_j such that $x_jq_n \notin E(H)$ and $x_{j+1}q_n \in E(H)$. Now $(x_j, x_{j+1}) \to (x_j, q_n)$ according to $\{13, 23\} \in \mathcal{F}$ and hence we obtain a shorter circuit $(x_0, x_1), (x_1, x_2), \ldots, (x_{j-1}, x_j), (x_j, q_n), (q_n, x_0)$ in $D \cup S$. This implies that $(x_{n-1}, x_n) \to (q_n, x_n)$. However $(x_n, q_n) \to (x_n, x_1)$ according to $\{12\}$ and hence we obatin a shorter circuit in $S \cup D$. Analogously if some pair (x, y) dominates (x_n, x_0) according to $\{23\}$ we arrive at a contradiction. Note that no pair (x, y) dominates (x_n, x_0) according to $\{13\}$.

Thus we continue by assuming that none of the $\{12\}, \{23\}, \{13\}$ belongs to $\mathcal{F}$ but one of the $\{12, 13\}, \{12, 23\} \in \mathcal{F}$.

First, suppose $\{12, 23\} \in \mathcal{F}$. Observe that $x_ix_{i+1} \in E(H)$, $0 \leq i \leq n$ (otherwise according to the priorities of the pairs there must be some vertex q_i such that $q_ix_i, q_ix_{i+1} \in E(H)$ and $(x_i, q_i) \in D$ and $(x_i, q_i) \to (x_i, x_{i+1}) \to (q_i, x_{i+1})$. However $(x_i, q_i) \to (x_{i+1}, q_i)$ according to $\{12, 23\}$ a contradiction). Thus by Claim 2.8 $x_0, x_1, \ldots, x_n$ induce a clique. Suppose $(u, v) \in S$, and $(u, v) \to (x_n, x_0)$ according to $\{13, 23\}$. This means there is q_n such that $q_nx_0 \notin E(H)$ and $q_nx_n \in E(H)$ and $(q_n, x_0) \to (x_n, x_0)$. Now according to $\{13, 23\}$, $(x_n, x_0) \to (q_n, x_0) \to (q_n, x_n)$ and hence both (x_n, q_n) and (q_n, x_n) are in $D \cup S$, a contradiction. Thus we may assume there is some pair $(x, y) \in S$ dominates (x_n, x_0) according to $\{12, 23\}$. Either there exists q_0 such that x_0q_0 is an edge, $x_nq_0 \notin E(H)$ and $(q_0, x_0) \in S$ or there exists q_n such that $q_nx_n \in E(H)$, $q_nx_0 \notin E(H)$ and $(x_n, q_n) \to (x_n, x_0)$. If the first case happens then according to $\{13, 23\}$, $(q_0, x_0) \to (q_0, x_n) \to (x_0, x_n)$, implying that $(x_0, x_n) \in S \cup D$. In the former case $x_{n-1}q_n \in E(H)$ as otherwise $(x_{n-1}, x_n) \to (x_n, q_n)$ a contradiction. However there exists some j such that $x_jq_n \notin E(H)$ and $x_{j+1}q_n \in E(H)$. Now $(x_j, x_{j+1}) \to (q_n, x_{j+1})$ according to $\{12, 23\}$ and $(x_j, x_{j+1}) \to (x_j, q_n) \to (x_{j+1}, q_n)$ according to $\{13, 23\}$. Therefore $(x_{j+1}, q_n), (q_n, x_{j+1}) \in D \cup S$, a contradiction.

Second, suppose $\{12, 13\} \in \mathcal{F}$. We show that x_ix_{i+1}, $0 \leq i \leq n$ is not an edge. For a contradiction suppose x_ix_{i+1} is not an edge. Now there exists q_i such that $q_ix_i, q_ix_{i+1} \in E(H)$ and $(q_i, x_{i+1}) \to (x_i, x_{i+1})$ according to $\{12, 13\}$ or $(x_i, q_i) \to (x_i, x_{i+1})$ according to $\{13, 23\}$. In any case we have $(x_i, q_i), (x_i, x_{i+1}), (q_i, x_{i+1}) \in D \cup S$. Now $q_ix_j \notin E(H)$, $j \neq i, i+1$ as otherwise when $x_{i+1}x_j \notin E(H)$, $(q_i, x_{i+1}) \to (x_j, x_{i+1})$ according to $\{12, 13\}$ and when $x_{i+1}x_j$ is an edge $x_ix_j \notin E(H)$

otherwise $(x_i, x_{i+1}) \rightarrow (x_j, x_{i+1})$ and hence $(x_i, q_i) \rightarrow (x_i, x_j)$, a shorter circuit. Note that when $x_j x_{j+1} \notin E(H)$, $j \neq i$ then $q_i q_j \notin E(H)$ as otherwise we obtain an induced C_4. Now we obtain an induced cycle of length more than 3 using $x_n x_0$ and x_i, q_i, x_{i+1}'s, a contradiction.

Therefore $x_0, x_1, \ldots, x_n$ induce a clique according to Claim 2.8. First suppose $(u, v) \rightarrow (x_n, x_0)$ according to $\{13, 23\}$. In this case there exists q_n such that $q_n x_n \in E(H)$ and $x_0 q_n \notin E(H)$ and $(q_n, x_0) \rightarrow (x_n, x_0)$. Now $q_n x_j \notin E(H)$, $j \neq 0, n$ as otherwise $(q_n, x_0) \rightarrow (x_j, x_0)$ and hence we obtain a shorter circuit $(x_0, x_1), (x_1, x_2), \ldots,$
$(x_{j-1}, x_j), (x_j, x_0)$ in $D \cup S$. However $(x_n, x_0) \rightarrow (q_n, x_0) \rightarrow (q_n, x_n)$ and hence both (x_n, q_n) and (q_n, x_n) are in $D \cup S$, according to $\{12, 13\}$, a contradiction. Therefore we assume that (x_n, x_0) is dominated by a pair according to $\{12, 13\}$ and analogously we arrive at a contradiction.

Second, suppose $x_0 x_n \notin E(H)$.

Claim 2.9 $\{13, 23\}$ *is not the only forbidden pattern in $\mathcal{F}$.*

Proof: For contradiction suppose $\{13, 23\}$ is the only forbidden pattern. Note that by assumption there exists at least one $0 \leq r \leq n$ such that $x_r x_{r+1}$ is not an edge of H (in particular $r = n$, the indexes are modul $n+1$). Now according to the rules of the algorithm the assumption that $\{13, 23\}$ is the only forbidden pattern, if $x_i x_{i+1} \notin E(H)$, $0 \leq i \leq n$ there must be a vertex p_i such that $x_i p_i, x_i p_{i+1} \in E(H)$ and $(x_i, p_i) \in S \cup D$ dominates (x_i, x_{i+1}). Moreover $(x_i, x_{i+1}) \rightarrow (p_i, x_{i+1})$ and hence $(p_i, x_{i+1}) \in S \cup D$. We may assume circuit C has the minimum number of pairs (x_i, x_{i+1}) that $x_i x_{i+1}$ is not an edge.

Observation : If $x_j x_{j+1}, x_{j+1} x_{j+2}$ are edges of H then $x_j x_{j+2}$ is also an edge of H as otherwise $(x_j, x_{j+1}) \rightarrow (x_j, x_{j+2})$ and hence $(x_j, x_{j+2}) \in S \cup D$, implying a shorter circuit. If $x_j x_{j+1}$ is not an edge of H then for every $j' \neq j, j+1$ at most one of the $x_j x_{j'}, x_{j+1} x_{j'}$ is an edge of H as otherwise $(x_j, x_{j+1}) \rightarrow (x_{j'}, x_{j+1})$ and hence we have a shorter circuit in $S \cup D$.

First, suppose $x_{r+1} x_{r+2}$ is an edge of H. By observation above $x_r x_{r+2} \notin E(H)$ and hence $p_r x_{r+2} \notin E(H)$ as otherwise $(x_r, p_r) \rightarrow (x_r, x_{r+2})$ and hence $(x_r, x_{r+2}) \in S \cup D$, a shorter circuit in $S \cup D$. If $x_{r+2} x_{r+3}$ is also an edge of H then by Observation above $x_{r+1} x_{r+3} \in E(H)$ and hence $x_r x_{r+3} \notin E(H)$. Now $p_r x_{r+3} \notin E(H)$ as otherwise $(x_r, p_r) \rightarrow (x_r, x_{i+r})$, implying a shorter circuit in $S \cup D$.

Second, let j be the first index after r (in the clockwise direction) such that $x_j x_{j+1}$ is not an edge of H. By the above Observation we may assume that $x_{r+1} x_j$, $r + 1 \neq j$ is an edge of H and none of $x_r x_j, x_r x_{j+1}, x_{r+1} x_{j+1}$ is an edge of H. Now $p_j x_r \notin E(H)$ as otherwise $(x_j, p_j) \rightarrow (x_j, x_r)$ and hence $(x_j, x_r) \in S \cup D$, implying a shorter circuit in $S \cup D$. Now $x_{j+1} x_r \notin E(H)$ as otherwise $(p_j, x_{j+1}) \rightarrow (p_j, x_r) \rightarrow (x_{j+1}, x_r)$ and hence $(x_{j+1}, x_r) \in S \cup D$, a shorter circuit in $S \cup D$. Now $p_r x_{j+1} \notin E(H)$ as otherwise $(x_r, p_r) \rightarrow (x_r, x_{j+1})$ and hence we obtain a shorter circuit $(x_0, x_1), (x_1, x_2), \ldots, (x_{r-1}, x_r), (x_r, x_j), \ldots, (x_n, x_0) \in S \cup D$. Similarly $p_r x_j \notin E(H)$. as otherwise we obtain a shorter circuit in $S \cup D$. Moreover $p_r p_j \notin E(H)$ as otherwise $(x_r, p_r) \rightarrow (x_r, p_j)$ and by replacing x_j with p_{r+1} we obtain circuit $C_1 = (x_0, x_1), \ldots, (x_{r-1}, x_r), (x_r, p_j), (p_j, x_{j+1}), (x_{j+1}, x_{r+2}), \ldots, (x_n, x_0)$, and since $p_{r+1} x_{r+2}$ is an edge of H, C_1 contradicts the our assumption about C (if $j \neq r + 1$ then $(p_r, x_{r+1}) \rightarrow (p_r, x_j) \rightarrow (x_{r+1}, x_j)$ and hence $(x_{r+1}, x_j) \in S \cup D$, a shorter circuit in $S \cup D$).

By applying two above arguments we conclude that p_r is not adjacent to any x_j, $j \neq r, r+1$ and p_r is not adjacent to any p_j. Therefore we obtain an induced cycle $x_r, p_r, x_{r+1}, x_j, p_j, x_{j+1}, \ldots, x_r$

of length more than 3, and by Claim 2.5 we conclude that there exists a circuit in a strong component of H^+. $\diamond$

Claim 2.10 *If one of the patterns $\{12\}, \{23\}, \{13\}$ is in $\mathcal{F}$, then $x_0, x_1, \ldots, x_n$ induce an independent set.*

Proof: For a contradiction suppose $x_i x_j \in E(H)$. Now $x_{i-1}x_i \in E(H)$ as otherwise $(x_{i-1}, x_i) \to (x_j, x_i)$ when $x_j x_{i-1} \in E(H)$ according to $\{13, 23\}$ and when $x_{i-1}x_j \notin E(H)$ then $(x_{i-1}, x_i) \to (x_j, x_i)$ when $\{23\} \in \mathcal{F}$ or $(x_{i-1}, x_i) \to (x_{i-1}, x_j)$ when $\{23\} \in \mathcal{F}$) or $(x_{i-1}, x_i) \to (x_{i-1}, x_j)$ when $\{23\} \in \mathcal{F}$). In any case we have a shorter circuit. Therefore $x_{i-1}x_i \in E(H)$ and by continuing this argument we conclude that $x_0 x_n \in E(H)$, contradicting our assumption. $\diamond$

Since $x_0 x_n$ is not an edge up to symmetry and using the same argument in the complement of H in the remaining we just consider the cases $\{13\} \in \mathcal{F}$ and $\{12, 23\} \in \mathcal{F}$.

We first consider $\{13\} \in \mathcal{F}$. Suppose (x_n, x_0) is dominated by some pair $(x, y) \in S$ according to forbidden pattern $\{13, 23\}$. This means that there exists some q_n such that $(x_n, q_n) \in S$ dominates (x_n, x_0), $q_n x_0, q_n x_n \in E(H)$. Now $x_{n-1}q_n \in E(H)$ as otherwise according to forbidden pattern $\{13\}$, $(x_{n-1}, x_n) \to (x_{n-1}, q_n)$ and $(x_{n-1}, q_n) \to (x_{n-1}, x_0)$ a shorter circuit in $D \cup S$. However according to $\{13, 23\}$, $(x_n, q_n) \to (x_n, x_{n-1})$, a contradiction. Therefore (x_n, x_0) is dominated by a pair $(x, y) \in S$ according to pattern $\{13\}$. This means there exists q_0 such that $q_0 x_0 \in E(H)$, $q_0 x_n \notin E(H)$ and $(x_n, q_0) \in S$ dominates (x_n, x_0). Consider an edge $x_i q_i$ for some $0 \leq i \leq n$. By applying similar argument in the beginning of the case, $q_i x_{i+1} \notin E(H)$. We show that q_i is not adjacent to any other x_j, $j \neq i, i+1$. Otherwise

$(x_i, x_{i+1}) \to (q_i, x_{i+1})$ and $(q_i, x_{i+1}) \to (x_j, x_{i+1})$ and hence we obtain a shorter circuit. Moreover we show that x_i can be replaced by x_i and obtain a circuit of length n as follows: $(x_{i-1}, x_i) \to (x_{i-1}, q_i)$ and $(x_i, x_{i+1}) \to (q_i, x_{i+1})$ and hence

$$(x_0, x_1), (x_1, x_2), \ldots, (x_{i-2}, x_{i-1}), (x_{i-1}, q_i), (q_i, x_{i+1}), (x_{i+1}, x_{i+2}), \ldots, (x_{n-1}, x_n), (x_n, x_0)$$

is also a circuit in $D \cup S$.

Thus $(q_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, q_0)$ is also a circuit in $D \cup S$. Moreover if q_0 has a neighbor then this neighbor is not adjacent to any of the x_j, $j \neq 0$. This give rise to a disjoint connected components $H_0, H_1, \ldots, H_n$ where there is no edge between H_i, H_j, $i \neq j$. Moreover (x_i, x_j) and (x_r, x_s), for $(i, j) \neq (r, s)$ are in different components of H^+. These imply that (x_0, x_n) is also green and by adding (x_0, x_n) into D we don't form a circuit as otherwise there would be an earlier circuit in D.

Second we consider $\{12, 23\} \in \mathcal{F}$. Since $x_n x_0$ is not an edge, and (x_n, x_0) is in a non-trivial component, (x_n, x_0) is dominated by some pair $(x, y) \in S$ according to $\{13, 23\}$. This means that there exists some q_n such that $(x_n, q_n) \in S$ dominates (x_n, x_0), $q_n x_0, q_n x_n \in E(H)$. Note that $(x_n, q_n) \to (x_n, x_0) \to (q_n, x_0)$. However $(x_n, q_n) \to (x_0, q_n)$ according to $\{12, 23\}$, a contradiction.

Case 2. $\{12, 23\} \in \mathcal{F}$, and $\{13, 23\}, \{12, 13\} \notin \mathcal{F}$.

First, suppose $x_n x_0$ is an edge of H.

We show that $x_{n-1}x_n$ is an edge of H. Otherwise according to the Algorithm and the assumption that none of the $\{13, 23\}, \{12, 13\}$ is in $\mathcal{F}$ then one of the patterns $\{12\}$, $\{23\}$, $\{13\}$ is in $\mathcal{F}$. Now $x_{n-1}x_0$ is not an edge as otherwise $(x_n, x_0) \to (x_{n-1}, x_0)$ and hence we have a shorter circuit. However $(x_{n-1}, x_n) \to (x_0, x_n)$ when $\{23\} \in \mathcal{F}$ and $(x_n, x_0) \to (x_{n-1}, x_0)$ when $\{12\} \in \mathcal{F}$ and $(x_{n-1}, x_n) \to (x_{n-1}, x_0)$ when $\{13\} \in \mathcal{F}$. In any case we obtain a shorter circuit. Thus $x_{n-1}x_n$ is an edge and by applying the previous argument we conclude that $x_i x_{i+1}$, $0 \le i \le n$ is an edge and hence $x_0, x_1, \ldots, x_n$ induce a clique.

Note that (x_0, x_n) does not dominate any pair according to one of the patterns $\{12\}$, $\{23\}$, $\{13\}$. For contradiction suppose (x_n, x_0) is dominated by some pair $(x, y) \in S$ according to $\{23\}$ (the argument of other cases is analogous). This means there exists w such that $(w, x_0) \to (x_n, x_0) \to (x_n, w)$. Now if $wx_j \notin E(H)$ then $(w, x_0) \to (x_j, x_0)$ and we obtain a shorter circuit. Now $(x_{n-1}, x_n) \to (x_{n-1}, w) \to (x_{n-1}, x_0)$ implying a shorter circuit in $D \cup S$. Therefore we may assume that $\{12, 23\}$ is the only forbidden pattern in $\mathcal{F}$.

Since (x_n, x_0) dominates a pair in S because of symmetry we continue by assuming there exists q_n such that $x_n q_n \in E(H)$ and $x_0 q_n \notin E(H)$. In this case $(x_0, x_n)(q_n, x_n)$ is an edge of H^+ (symmetric). Now $(x_{n-1}, x_n) \to (q_n, x_n)$ and hence $(q_n, x_n) \in D \cup S$. On the other hand we have $(x_n, q_n) \in S$, implying a circuit in $D \cup S$, a contradiction.

Second, suppose $x_n x_0$ is not an edge of H.

According to the algorithm we may assume one of the $\{12\}, \{23\}, \{13\}$ is in $\mathcal{F}$. Now by the same argument as in Claim 2.10, $x_0, x_1, \ldots, x_n$ induce an independent set. By using symmetry and applying argument in Case B we conclude (x_0, x_n) can be added into D without creating a circuit.

This completes the correctness proof of the algorithm and hence the proof of Theorem 2.3. ⋄

Corollary 2.11 *Each problem* ORD($\mathcal{F}$) *with* $\mathcal{F} \in \mathcal{F}_3$ *can be solved in polynomial time.*

Remark. Our algorithm is linear in the size of H^+. The number of edges in H^+ is at most n^3 since each pair (x, y) has at most n out-neighbors. Thus the algorithms runs in $O(n^3)$, where $n = |V(H)|$. In some cases, e.g., when $|\mathcal{F}| = 1$, this can be improved to $O(nm)$, where $m = |E(H)|$.

2.1 Obstruction Characterizations

Many of the known graph classes discussed here have obstruction characterizations, usually in terms of forbidden induced subgraphs or some other forbidden substructures. A typical example is chordal graphs, whose very definition is a forbidden induced subgraph description: no induced cycles of length greater than three. Interval graphs have been characterized by Lekkerkerker and Boland [19] as not having an induced cycle of length greater than three, and no substructure called an asteroidal triple. Proper interval graphs have been characterized by the absence of induced cycles of length greater than three, and three special graphs usually called net, tent, and claw [27]. Comparability graphs have a similar forbidden substructure characterization [11].

The constraint digraph offers a natural way to define a common obstruction characterization for all these graph classes. In fact, Theorem 2.3 can be viewed as an obstruction characterization of ORD($\mathcal{F}$) for any $\mathcal{F} \in \mathcal{F}_3$, i.e., each of these classes is characterized by the absence of a circuit in a strong component of the constraint digraph. Moreover, our algorithm is a certifying algorithm, in the sense that when it fails, it identifies a circuit in a strong component of H^+.

For some of the sets $\mathcal{F} \in \mathcal{F}_3$, we have an even simpler forbidden substructure characterization. We say x, y is an *invertible pair* of H if (x, y) and (y, x) belong to the same strong component of H^+. We say $\mathcal{F}$ is *nice* if it is one of the following sets

$$\{\{13\}\}, \{\{12, 23\}\}, \{\{13\}, \{13, 23\}\}, \{\{13\}, \{12, 13\}, \{13, 23\}\}.$$

By following the correctness proof of our algorithm, it is not difficult to see that if $\mathcal{F}$ is nice, then the algorithm does not create a circuit as long as every strong component S of H^+ has $S \cap \overline{S} = \emptyset$. Thus we obtain the following theorem for nice sets $\mathcal{F}$.

Theorem 2.12 *Suppose $\mathcal{F}$ is nice. A graph H admits an $\mathcal{F}$-free ordering if an only if it does not have an invertible pair.* $\diamond$

In fact the correctness proof shows that if there is any circuit in a strong component of H^+, then there is also a circuit of length two.

Theorem 2.12 applies to, amongst others, interval graphs, proper interval graphs, comparability graphs and co-comparability graphs.

3 Bipartite graphs

In this section we consider bipartite graphs H with a fixed bipartition $U \cup V$. We also denote by $\mathcal{B}_k$ the collection of sets $\mathcal{F}$ consisting of bipartite patterns with k vertices each. We prove that BiOrd$_4$ is polynomial-time solvable, and so BiOrd($\mathcal{F}$) is polynomial-time solvable for each $\mathcal{F} \in \mathcal{B}_4$. Each forbidden pattern $F \in \mathcal{F}$ imposes constraints for those 4-tuples of vertices that induce a subgraph isomorphic to $\mathcal{F}$. We construct an auxiliary digraph H^+, also called a *constraint digraph*. The vertex set of H^+ consists of the pairs $(x, y) \in (U \times U) \cup (V \times V)$, where $x \neq y$, and the arc-set of H^+ is defined as follows.

There is an arc from (x, y) to (z, y) and an arc from (y, z) to (y, x) whenever the vertices x, y, z from the same part (U or V) of the bipartition, ordered $x < y < z$, together with some vertex v from the other part of the bipartition (V or U), induce a forbidden pattern in $\mathcal{F}$. There is also an arc from (x, y) to (u, v) and an arc from (v, u) to (y, x) whenever the vertices x, y from the same part, ordered as $x < y$, together with some vertices u, v from the other part, ordered as $u < v$, induce a pattern in $\mathcal{F}$.

We say that a pair (x, y) *dominates* (x', y') and we write $(x, y) \to (x', y')$ if there is an arc from (x, y) to (x', y') in H^+.

A *circuit* in a subset D of H^+ is a sequence of pairs $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$, $n \geq 1$, that all belong to D. Observe that $x_0, x_1, \ldots, x_n$ belong to the same bipartition part of $V(H)$.

Ordering with bipartite forbidden 4-patterns, BiOrd$_4$
Input: A bigraph $H = (U, V)$ and a set $\mathcal{F} \subseteq \mathcal{B}_4$ of bipartite forbidden patterns on four vertices

Output: And ordering of the vertices in U and an ordering of the vertices in V that is a $\mathcal{F}$-free ordering or report that there is no such ordering.

Algorithm for BiOrd$_4$

1. If a strong component S of H^+ contains a circuit then report that no solution exists and exit. Otherwise, remove $\emptyset$, $\{11', 12', 21', 22'\}$ and $\{11', 21', 31'\}$ from $\mathcal{F}$. If $\mathcal{F}$ is empty after this step, then return any ordering of vertices of H and stop.

2. Set D to be the empty set.

3. Choose a strong component S of H^+ that is green with respect to D. The choice is made according to the following rules.

   a) If $\mathcal{F}$ contains one of the forbidden patterns $\{11', 12', 21'\}, \{12', 21', 22'\}, \{11', 12', 22'\}$, $\{11', 21', 22'\}$ then priority is given to a component S containing (x, y) where x, y have a common neighbor in H. If there is a choice then it is preferred S to be a trivial component. Subject to these preferences, if there are several candidates, then priority is given to the ones that are sink components in H^+.

   b) If $\mathcal{F}$ contains one of the forbidden patterns $\{11'\}, \{22'\}), \{12'\}, \{21'\}$ then priority is given to a component S containing (x, y) where x, y have a common non-neighbor in H. If there is a choice then it is preferred S to be a trivial component. Subject to these preferences, if there are several candidates, then priority is given to the ones that are sink components in H^+.

4. If by adding S into D we do not close a circuit, then we add S into D and discard $\overline{S}$. Otherwise we add $\overline{S}$ and its outsection (all vertices in H^+ that are reachable from $\overline{S}$) into D and discard S and its insection (the vertices that can reach S). Return to Step 3 if there are some strong components of H^+ left.

5. For every $(x, y) \in D$, place x before y in the final ordering.

A polynomial-time solution to BiOrd$_4$ is implicit in the following main result of this section.

Theorem 3.1 *Let $\mathcal{F} \in \mathcal{B}_4$ and let H^+ be the constraint digraph of H with respect to $\mathcal{F}$. Then H has a $\mathcal{F}$-free ordering of its parts if and only if no strong component of H^+ contains a circuit.*

Proof: The argument for showing that the algorithm does not create a circuit is very similar to the proof of Lemma 2.7. However for the sake of completeness we present the case when $\mathcal{F} = \{\{12', 21'\}, \{12', 21', 22'\}\}$. We claim that the algorithm will never create a circuit, and hence yield the desire ordering. We also prove an stronger version by assuming that H does not have an invertible pair. Then the components of H^+ come in conjugate pairs $S, \overline{S}$. We claim that the Algorithm for BiOrd$_4$ does not creates a circuit.

Otherwise, suppose the addition of S creates circuits for the first time, and $(x_0, x_1), (x_1, x_2)$, $\dots, (x_n, x_0)$ is a shortest circuit created at that time. Note that $n > 1$, according to the rules of the algorithm. Without loss of generality, assume that the pair (x_n, x_0) (and possibly other pairs) lies in S.

We may assume that (x_n, x_0) is in a non-trivial component S of H^+, otherwise (x_n, x_0) or (x_0, x_n) is a sink, and clearly no circuit is created during the preliminary stage when sinks were handled. Thus (x_n, x_0) dominates and is dominated by some pair, which may be assumed to be the same pair, say (y_n, y_0). We claim that each x_i has a private neighbor y_i but not to any other y_j with $j \neq i$.

Now $y_n x_j \notin E(H)$ for $j \neq n, 0$, as otherwise (y_n, y_0) dominates (x_j, x_0) and hence $(x_j, x_0) \in D \cup S$ implying a shorter circuit in $D \cup S$. Moreover $y_0 x_j$, $j \neq n, 0$ as otherwise (y_n, y_0) dominates (x_n, x_j) and hence $(x_n, x_j) \in D \cup S$, again implying a shorter circuit in $D \cup S$. For summary we have :

- $x_0 y_0, x_n y_0 \in E(H)$
- $x_j y_n \notin E(H)$, $j \neq n$
- $x_j y_0 \notin E(H)$, $j \neq 0$.

If (x_0, x_1) does not dominates any pair in H^+ then we show that (x_n, x_1) does not dominates any pair in H^+ either, and hence by the rules of the algorithm (x_n, x_1) is in D already, contradicting the minimality of n. In contrary suppose (x_0, x_1) does not dominate any pair and (x_n, x_1) dominates some pair (y_n, y_1) in H^+. Note that $x_1 y_n$ is not an edge. Now $x_0 y_1$ is an edge and hence (x_n, x_0) dominate $(y_n, y_1) \in D \cup S$. Moreover (y_n, y_1) dominates (x_n, x_1) and hence $(x_n, x_1) \in D \cup S$ implying a shorter circuit in $D \cup S$.

Therefore (x_0, x_1) dominates $(y_0, y_1) \in D \cup S$. Recall that none of the $x_n y_0, x_0 y_n, x_1 y_0, x_1 y_n$ is an edge of H. Now $x_n y_1 \notin E(H)$ as otherwise (x_n, x_0) dominates (y_1, y_0) and hence $(y_1, y_0) \in D \cup S$. Now (y_1, y_0) dominates (x_1, x_0) implying that $(x_1, x_0) \in D \cup S$ and hence yielding a shorter circuit in $D \cup S$. We conclude that (x_0, x_1) and (x_n, x_1) both are in non trivial components of H^+, and $x_n y_n, x_0 y_0, x_1 y_1$ are independent edges. By continuing this argument we conclude that there are independent edges $x_0 y_0, x_1 y_1, ..., x_n y_n$.

Let X denotes the set of vertices of H adjacent to all x_i, or to all y_i. Note that X is complete bipartite, as otherwise H contains a six-cycle (with diametrically opposite vertices in X), and hence an invertible pair. Now we claim that any vertex v not in X is adjacent to at most one x_i (or y_i). Otherwise we may assume v is not adjacent to x_{i-1} but is adjacent to x_i and x_j, which would imply that (x_{i-1}, x_j) is in the same component as (x_{i-1}, x_i), contradicting the minimality of our circuit. (Thus it is impossible to have a path of length two between x_i and $x_j (i \neq j)$ without going through X.) More generally, we can apply the same argument to conclude that any path between x_i and $x_j (i \neq j)$ must contain a vertex of X. Therefore, $H - X$ has distinct components $H_1, H_2, \dots H_n$, where H_i contains x_i and y_i. We claim that the component of H^+ containing the pair (x_i, x_{i+1}) consists of all pairs (u, v) where u is in H_i and v is in H_{i+1}. This implies that S does not contain any (x_i, x_{i+1}) other that (x_n, x_0); it also implies that both S and $\overline{S}$ are green.

Now consider the addition of $\overline{S}$. If it also leads to a circuit $(z_0, z_1), \dots, (z_r, z_0)$, we may assume that $z_0 = x_n$ and $z_r = x_0$. By a similar argument we see that $\overline{S}$ does not contain any other (z_i, z_{i+1}). This means that $(x_0, x_1), \dots, (x_{n-1}, x_n) = (x_{n-1}, z_0), (z_0, z_1), \dots (z_r, z_0) = (z_r, x_0)$ was an earlier circuit, contradicting the assumption.

This completes the proof of Theorem 3.1 $\diamond$

Corollary 3.2 *Each problem* BIORD($\mathcal{F}$) *with* $\mathcal{F} \in \mathcal{F}_4$ *can be solved in polynomial time.*

3.1 Obstruction Characterizations

It is similarly the case that the constraint digraph offers a unifying concept of an obstruction for graph classes BIORD($\mathcal{F}$), $\mathcal{F} \in \mathcal{B}_4$. Namely, Theorem 3.1 characterizes all these classes by the absence of a circuit in a strong component of the constraint digraph. In some cases we can again simplify the obstructions to a bipartite version of invertible pairs.

An *invertible pair* of H is a pair of vertices u, v from the same part of the bipartition such that both (u, v) and (v, u) lie on the same directed cycle of H^+. Thus a circuit of length two in a strong component of H^+ corresponds precisely to an invertible pair.

For our first illustration we discuss the case of co-circular-arc bigraphs. A *co-circular-arc bigraph* is a bipartite graph whose complement is a circular arc graph. A complex characterization of co-circular-arc graphs by seven infinite families of forbidden induced subgraphs has been given in [25], later simplified to a Lekerkerker-Boland-like characterization by forbidden induced cycles and edge asteroids in [8]. These graphs seem to be the bipartite analogues of interval graphs, see [8]. One reason may be that co-circular-arc bigraphs are precisely the intersection graphs of 2-directional rays [22].

We observe the following simple characterization.

Theorem 3.3 *Let* $H = (B, W)$ *be a bipartite graph. Then the following are equivalent.*

- *H is a co-circular-arc bigraph.*
- *H admits an $\mathcal{F}$-free ordering where* $\mathcal{F} = \{\{12', 21'\}, \{12', 21', 22'\}\}$.
- *H has no invertible pair.*

Proof: It was shown in [18] that (1) and (2) are equivalent. (In [18] $\mathcal{F}$-free orderings are described by an equivalent notion of so-called min orderings.) According to proof of Theorem 3.1 for $\mathcal{F}$ we assume that H does not have an invertible pair. Therefore (2) and (3) are equivalent and hence the theorem is proved. ◇

A bipartite graph $G = (V, U)$ is called *proper interval bigraph* if the vertices in each part can be represented by an inclusion-free family of intervals, and a vertex from V is adjacent to a vertex from U if and only if their intervals intersect. They are also known as bipartite permutation graphs [14, 23, 24].

Theorem 3.4 *Let* $H = (B, W)$ *be a bipartite graph. Then the following are equivalent.*

- *H is a proper interval bigraph*
- *H admits an $\mathcal{F}$-free ordering where* $\mathcal{F} = \{\{12', 21'\}, \{12', 21', 22'\}, \{11', 12', 21'\}\}$.
- *H does not have an invertible pair.*

Proof: It was noted in [14] that H admits an $\mathcal{F}$-free ordering if and only if H is a bipartite permutation graph (proper interval bigraph). (In [14] $\mathcal{F}$-free orderings are described by an equivalent notion of min-max orderings.) Therefore (1) and (2) are equivalent. It is easy to see that if no strong component of H^+ contains a circuit of length two, i.e. if H has no invertible pair, then the Algorithm for BiOrd_4 does not create a circuit. Therefore (2) and (3) are equivalent and hence the theorem is proved. $\diamond$

4 Remarks and conclusions

As noted earlier, Duffus, Ginn, and Rodl have found many examples of NP-complete problems $\text{Ord}(\mathcal{F})$; in fact if $\mathcal{F}$ consists of a single ordered pattern, they offered strong evidence that $\text{Ord}(\mathcal{F})$ is NP-complete as soon as the pattern is 2-connected. We offer just two simple examples to illustrate some NP-complete cases.

Proposition 4.1 *For every $k \geq 4$ there exists a set $\mathcal{F} \in \mathcal{F}_k$ such that* $\text{Ord}(\mathcal{F})$ *is NP-complete.*

Proof: We show that if $\mathcal{F}$ is a set of all forbidden patterns on k vertices where each of them contains $\{12, 23, 34, \dots, (k-1)k\}$ as a subset, then Ord_k is NP-complete. We reduce the problem to $(k-1)$-colorability. Let H be an arbitrary graph. If H is $(k-1)$-colorable with color classes $X_1, X_2, \dots, X_{k-1}$, then we put all the vertices in X_i before all the vertices in X_{i+1}, $1 \leq i \leq k-2$. This way we obtain an ordering of the vertices and it is clear that it does not contain any of the forbidden patterns in $\mathcal{F}$.

Now suppose there is an ordering $v_1, v_2, \dots, v_n$ of the vertices in H without seeing any forbidden pattern in $\mathcal{F}$. Let X_1 be the set of vertices v_j, $1 \leq j \leq n$ that have no neighbor before v_j. Now for every $2 \leq i \leq k-1$, let X_i be the set of vertices v_j, $1 \leq j \leq n$ from set $V(H) \setminus (\cup_{\ell=1}^{\ell=i-1} X_\ell)$ that have no neighbor before v_j. Note that by definition each X_i, $1 \leq i \leq k-1$ is an independent subset of H. Moreover $V(H) = \cup_{\ell=1}^{\ell=i-1} X_\ell$ as otherwise we obtain k vertices $u_1 < u_2 \cdots < u_k$ where $u_j u_{j+1}$, $1 \leq j \leq k-1$ is an edge of H and hence we find a forbidden pattern from $\mathcal{F}$. Thus H is $(k-1)$-colorable. $\diamond$

We note that in Damaschke's paper [5] the complexity of $\text{Ord}(\mathcal{F})$ was left open for $\mathcal{F} = \{12, 23, 34\}$. (However, other folklore solutions for this particular case have been reported since.)

In the case of bipartite graphs, we offer the following simple example.

Proposition 4.2 BiOrd_5 *is NP-complete for set* $\mathcal{F} = \{\{11', 31', 51'\}\}$.

Proof: Let M be a $m \times n$ matrix with entities 0 and 1. Finding an ordering of the columns such that in each row there are at most two sequences of consecutive 1's has been shown to be NP-complete in [12]. Now from an instance of a matrix M we construct a bipartite graph $H = (A, B, E)$ where A represents the set of columns and B represents the set of rows in M. There is an edge between $a \in A$ and $b \in B$ if the entry in M, corresponding to row a and column b is 1. Now if we were able to reorder to columns with the required property we would be able to find the ordering of H without seeing the forbidden pattern in F and vice versa. $\diamond$

There are natural polynomial problems $\text{Ord}(\mathcal{F})$ for sets $\mathcal{F}$ of larger patterns. For instance, strongly chordal graphs are characterized as $\text{Ord}(\mathcal{F}), \mathcal{F} \in \mathcal{F}_4$, in [7]. In fact, an algorithm similar to the one presented here can be developed for this case. (We will be happy to communicate the details to interested readers.)

There is a natural version of $\textsc{Ord}(\mathcal{F})$ for digraph patterns $\mathcal{F}$. We are given an input digraph H and a set $\mathcal{F}$ of forbidden digraph patterns (each digraph pattern is an ordered digraph). The decision problem asking whether an input digraph admits an ordering without forbidden patterns in $\mathcal{F}$ is denoted by $\textsc{DiOrd}(\mathcal{F})$. Let $\mathcal{D}_k$ denote the collection of sets $\mathcal{F}$ of digraph patterns with k vertices. The problem $\textsc{DiOrd}_k$ asks, for an input $\mathcal{F} \in \mathcal{D}_k$ and a digraph H, whether or not H has an $\mathcal{F}$-free ordering.

The algorithms in [9, 17] illustrate two cases where $\textsc{DiOrd}(\mathcal{F})$ problems have been solved by algorithms similar to the algorithm for $\textsc{Ord}_3$, and the obstructions characterized as invertible pairs. (The problems in [9, 17] are not presented as $\textsc{Ord}(\mathcal{F})$, but they can easily be so reformulated.) We believe many other digraph problems can be similarly handled. In fact we wonder whether the problem $\textsc{DiOrd}(\mathcal{F})$ is polynomial for every set $\mathcal{D} \in \mathcal{D}_3$.

We conjecture that for every set $\mathcal{F}$ of forbidden patterns, $\textsc{Ord}(\mathcal{F})$ is either polynomial or NP-complete.

References

[1] K.S. Booth and G.S. Lueker. *Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms*, Journal of Computer and System Sciences 13 (3) (1976) 335–379.

[2] A. Brandstädt, V.B. Le and J.Spinrad. Graph Classes: A Survey, SIAM Monographs on Discrete Mathematics and Applications (1999).

[3] T. Calamoneri, S. Caminiti, R. Petreschi, S. Olariu. *On the L(h,k)-labeling of co-comparability graphs and circular-arc graphs*, Networks 53(1) (2009) 27–34 .

[4] D.G. Corneil, S.Olariu, L.Stewart, *The LBFS Structure and Recognition of Interval Graphs*, SIAM J. Discrete Math. 23(4) : (2009) 1905-1953

[5] P. Damaschke, *Forbidden Ordered Subgraphs*, Topics in Combinatorics and Graph Theory. (1990) 219–229.

[6] D. Duffus, M. Ginn, V. Rödl *On the computational complexity of ordered subgraph recognition*, Random Structures and Algorithms. 7 (3) (1995)223-268.

[7] M.Farber, *Characterizations of strongly chordal graphs*, Discrete Mathematics 43(2-3) (1983) 173–189.

[8] T. Feder, P. Hell and J. Huang. *List homomorphisms and circular arc graphs*, Combinatorica 19 (1999) 487–505.

[9] T. Feder, P. Hell, J. Huang and A. Rafiey. *Interval graphs, adjusted interval graphs and reflexive list homomorphisms*, Discrete Appl. Math. 160 (2012) 697–707.

[10] D. R. Fulkerson and O. A. Gross. *Incidence matrices and interval graphs*, Pacific J. Math. (1965) 835–855.

[11] T. Gallai, *Transitiv orientierbare Graphen*, Acta Math. Acad. Sci. Hung 18 (1967) 25–66.

[12] P. W. Goldberg, M. C. Golumbic, H. Kaplan and R. Shamir. *Four strikes against physical mapping of DNA*, J. Comput. Biol. (1995) 139–152.

[13] M.C. Golumbic. Algorithmic Graph Theory and Perfect Graphs, Academic Press (1980).

[14] G. Gutin, P. Hell, A. Rafiey and A. Yeo, *A dichotomy for minimum cost graph homomorphisms*, European Journal of Combinatorics 29 (2008) 900–911.

[15] M. Habib, R. McConnell, Ch. Paul and L. Viennot, *Lex-BFS and partition refinement, with applications to transitive orientation, interval graph recognition, and consecutive ones testing*, Theor. Comput. Sci. 234 (2000) 59–84 .

[16] P. Hell and J. Huang, *Interval bigraphs and circular arc graphs*, Journal of Graph Theory 46 (2004) 313–327.

[17] P. Hell and A. Rafiey. *Monotone Proper Interval Digraphs*, SIAM J. Discrete Math. 26(4) (2012) 1576-1596.

[18] P. Hell, M. Mastrolilli, M.M.Nevisi and A. Rafiey. *Approximation of Minimum Cost Homomorphisms.* ESA 2012: 587–598.

[19] C.G. Lekkerkerker and J.C. Boland, *Representation of a finite graph by a set of intervals on the real line*, Fundamenta Math. 51 : 45-64 (1962).

[20] M.C. Lin, J.L. Szwarcfiter. *Characterizations and recognition of circular-arc graphs and subclasses: A survey*, Discrete Mathematics 309(18) (2009) 5618–5635.

[21] D. Rose, G. Lueker, R.E. Tarjan. *Algorithmic aspects of vertex elimination on graphs*, SIAM Journal on Computing 5 (2)(1976) 266–283.

[22] A.M.S. Shrestha, S. Tayu, S. Ueno. *On orthogonal ray graphs*, Discrete Applied Math. 158 (2010) 1650–1659.

[23] J.P. Spinrad, A. Brandstaedt, L. Stewart, *Bipartite permutation graphs*, Discrete Applied Math. 18 (1987) 279–292.

[24] J. Spinrad, *Efficient Graph Representations*, AMS, 2003.

[25] W. T. Trotter, J. Moore. *Characterization problems for graphs, partially ordered sets, lattices, and families of sets,* Discrete Math. 16 (1976), 361–381.

[26] W. T. Trotter. Combinatorics and Partially Ordered Sets–Dimension Theory, The Johns Hopkins University Press (1992).

[27] G. Wegner, *Eigenschaften der Nerven homologisch-einfacher Familien im* R^n, Ph.D. thesis, University of Göttingen (1967).