CMPT 981/409 Algebra and Computation
Homework 3
Due: Wednesday, October 28, 3:30pm

**Instructions:** You may work on and discuss homework problems with one another. Each student must individually write and submit their own solutions on Crowdmark. Be sure to list any collaborations in which you participated and any external sources you used. The use of generative AI in any form is prohibited.

1. (Matrix multiplication over $\mathbb{R}$ versus $\mathbb{C}$.) Recall that the matrix multiplication tensor $\langle n, n, n\rangle \in \mathbb{C}^{n^2 \times n^2 \times n^2}$ corresponds to the trilinear form

$$T(X, Y, Z) = \operatorname{trace}(XYZ^\top),$$

where $X$, $Y$, and $Z$ are $n \times n$ matrices of variables.

   (a) Suppose that $T$ has tensor rank $r$ over $\mathbb{C}$. That is, we can write $T$ as

$$T(X, Y, Z) = \sum_{i=1}^{r} a_i(X) b_i(Y) c_i(Z)$$

   where the linear forms $a_i(X)$, $b_i(Y)$, and $c_i(Z)$ have coefficients in $\mathbb{C}$.
   Prove that the tensor rank of $T$ over $\mathbb{R}$ is at most $4r$. In other words, show that $T$ can be written as

$$T(X, Y, Z) = \sum_{i=1}^{4r} \hat{a}_i(X) \hat{b}_i(Y) \hat{c}_i(Z)$$

   where the linear forms $a_i(X)$, $b_i(Y)$, and $c_i(Z)$ have coefficients in $\mathbb{R}$.

   (b) Suppose that for some $n \in \mathbb{N}$, the matrix multiplication tensor $\langle n, n, n\rangle$ has tensor rank $r$ over $\mathbb{C}$. Prove that there is a matrix multiplication algorithm with exponent $\log_n(r)$ that only uses constants from $\mathbb{R}$.

2. (Counting triangles and cliques.) Let $G = (V, E)$ be an undirected simple graph. A *triangle* in $G$ is a triple of vertices $a, b, c \in V$ such that $ab$, $bc$, and $ac$ are edges in $G$. More generally, a *$k$-clique* in $G$ is a $k$-tuple of vertices $a_1, \ldots, a_k$ such that $a_i a_j$ is an edge in $G$ for all $1 \le i, j \le k$ with $i \neq j$.

   (a) Describe and analyze an algorithm that receives an $n$-vertex graph $G$ as input and counts the number of triangles in $G$ in $O(n^\omega)$ time, where $\omega$ is the exponent of matrix multiplication.

   (b) Describe and analyze an algorithm that receives an $n$-vertex graph $G$ as input and counts the number of $k$-cliques in $G$ in $O(n^{\lceil \omega k/3 \rceil})$ time. For simplicity, you may assume that $k$ is a multiple of 3.

3. (Tensor rank of polynomial multiplication.) Let $\mu : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be the bilinear map that corresponds to multiplication of two polynomials of degree less than $n$ modulo $x^n$. In other words,

$$\mu(f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}) = (h_0, \ldots, h_{n-1}),$$

where $h(x) = \sum_{i=0}^{2n-2} h_i x^i$ is the product of $f(x) = \sum_{i=0}^{n-1} f_i x^i$ and $g(x) = \sum_{i=0}^{n-1}$. The corresponding structure tensor $T \in \mathbb{C}^{n \times n \times n}$ of $\mu$ is given by

$$T_{i,j,k} = \begin{cases} 1 & \text{if } i + j = k, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove that the rank of $T$ is at most $2n$.

(b) Prove that the border rank of $T$ is at most $n$.

4. (Practice with $p$-projections.) For $n \in \mathbb{N}$, let

$$\begin{aligned} \mathrm{Sum}_n(x_1, \ldots, x_n) &= x_1 + \cdots + x_n \\ \mathrm{Prod}_n(x_1, \ldots, x_n) &= x_1 \times \cdots \times x_n. \end{aligned}$$

Prove that the polynomial families $\mathrm{Sum} = (\mathrm{Sum}_n)_{n \in \mathbb{N}}$ and $\mathrm{Prod} = (\mathrm{Prod}_n)_{n \in \mathbb{N}}$ are not $p$-projections of one another.