CMPT 981/409 Algebra and Computation
Homework 4
Due: Friday, November 13, 2:30pm

Instructions: You may work on and discuss homework problems with one another. Each student must individually write and submit their own solutions on Crowdmark. Be sure to list any collaborations in which you participated and any external sources you used. The use of generative AI in any form is prohibited.

1. (Higher-order partial derivatives of arithmetic circuits.) Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial over $\mathbb{C}$ of degree d. Suppose that f can be computed by an arithmetic circuit of size s. For each $1 \leq i \leq n$ and $0 \leq k \leq d$, show that $\frac{\partial^k f}{\partial x_i^k}$, the k-th order partial derivative of f with respect to x_i, can be computed by an arithmetic circuit of size $\mathrm{poly}(s, d)$.

2. (Computing homogeneous parts more efficiently.) Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial over $\mathbb{C}$ of degree d. Write $f = f_0 + f_1 + \cdots + f_d$, where f_i is the degree-i homogeneous component of f. Suppose that f can be computed by an arithmetic circuit of size s. For each $0 \leq i \leq d$, show that f_i can be computed by an arithmetic circuit of size $O(sd)$. The circuit computing f_i need not be homogeneous.

   (Hint: expand $f(yx_1, yx_2, \ldots, yx_n)$ as a polynomial in y and interpolate.)

3. (Square roots of arithmetic circuits.) Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial over $\mathbb{C}$ of degree d. Suppose that f^2 can be computed by an arithmetic circuit of size s. Show that f can be computed by an arithmetic circuit of size $\mathrm{poly}(s, d)$.

   (Hint: expand $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$ as a power series. How much accuracy do you need?)

4. (Lower bounds for depth-three powering circuits.) In this exercise, we will prove lower bounds against *depth-three powering circuits*, a very restricted class of arithmetic circuits. For a polynomial f and a monomial $x_1^{a_1} \cdots x_n^{a_n}$, we write

$$\frac{\partial f}{\partial \vec{x}^{\vec{a}}} \coloneqq \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}} f$$

   for the partial derivative of f with respect to $\vec{x}^{\vec{a}}$. The *partial derivative dimension* of f, denoted $\dim \partial^{<\infty}(f)$, is defined as

$$\dim \partial^{<\infty}(f) \coloneqq \dim \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial f}{\partial \vec{x}^{\vec{a}}} : \vec{a} \in \mathbb{N}^n \right\}.$$

   We will see how $\dim \partial^{<\infty}(f)$ can be used to prove lower bounds against depth-three powering circuits.

   (a) Let $\ell(x_1, \ldots, x_n) = a_0 + a_1 x_1 + \cdots + a_n x_n$ be a polynomial of degree 1. If $\vec{x}^{\vec{a}}$ is a monomial of degree k, prove that the partial derivative $\frac{\partial \ell^d}{\partial \vec{x}^{\vec{a}}}$ is a scalar multiple of $\ell(x_1, \ldots, x_n)^{d-k}$.

   (b) Prove that $\dim \partial^{<\infty}(\ell(x_1, \ldots, x_n)^d) \leq d + 1$.

(c) Let f be a polynomial of the form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{s} \ell_i(x_1, \ldots, x_n)^d,$$

where each ℓ_i is a polynomial of degree 1. The polynomial f is said to have a depth-three powering circuit of size s. Prove that $\dim \partial^{<\infty}(f) \leq s(d+1)$.

(d) Prove that $\dim \partial^{<\infty}(x_1 \cdots x_n) = 2^n$. Conclude that if there is a representation of $x_1 \cdots x_n$ as

$$x_1 \cdots x_n = \sum_{i=1}^{s} \ell_i(x_1, \ldots, x_n)^n$$

where each ℓ_i is a polynomial of degree 1, then $s \geq \Omega(2^n/n)$.

(Not for submission: can you improve the lower bound to $\Omega(2^n/\sqrt{n})$?)