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Last lecture, we saw how to bound ω , the matrix multiplication exponent, by bounding the rank of the matrix multiplication tensor.

Today, we'll see how to bound the rank of the matrix multiplication tensor by instead bounding its border rank.

We'll also see how rectangular matrix multiplication can be useful for bounding ω .

Recall $T \in \mathbb{C}^{n \times m \times p}$ tensor

$$T(x, y)_k = \sum_{i, j} T_{ijk} x_i y_j \quad \text{bilinear form}$$

$$T(x, y)_k = \sum_{i=1}^r c_{i,k} a_i(x) b_i(y)$$

↑ bilinear algorithm of
rank r

$$R(T) := \text{rank}(T)$$

$$= \min \left\{ r : \exists \text{ rank-}r \text{ bilinear algorithm for } T \right\}$$

We can alternatively view a tensor as a trilinear form

$$\begin{aligned} T(x, y, z) &= \sum_{k=1}^p T(x, y)_k z_k \\ &= \sum_{i,j,k} T_{ijk} x_i y_j z_k. \end{aligned}$$

Bilinear algorithms have a more symmetric form when we work with tensors as trilinear forms.

If

$$T(x, y)_k = \sum_{i=1}^r c_{i,k} a_i(x) b_i(y),$$

then

$$\begin{aligned} T(x, y, z) &= \sum_k T(x, y)_k z_k \\ &= \sum_{i, k} c_{i,k} a_i(x) b_i(y) z_k \\ &= \sum_{i=1}^r a_i(x) b_i(y) \left(\sum_k c_{i,k} z_k \right) \\ &= \sum_{i=1}^r a_i(x) b_i(y) c_i(z). \end{aligned}$$

In this view, tensor rank is the minimum r st T can be written as $\sum_{i=1}^r a_i(x) b_i(y) c_i(z)$.

Let $\langle m, n, p \rangle \in \mathbb{C}^{mn \times np \times mp}$
be the tensor for $(m \times n) \times (n \times p)$
matrix multiplication.

Lemma. $R(\langle m, n, p \rangle) = R(\langle n, p, m \rangle)$
 $= R(\langle p, m, n \rangle).$

Proof let $T = \langle m, n, p \rangle \in \mathbb{C}^{mn \times np \times mp}$.

As a trilinear form,

$$\begin{aligned} T(X, Y, Z) &= \sum_{i,j,k} X_{ij} Y_{jk} Z_{ik} \\ &= \text{trace}(XYZ^T) \end{aligned}$$

Since $\text{trace}(AB) = \text{trace}(BA)$,

$$= \text{trace}(Z^T XY)$$

$$= S(Z^T, X, Y^T)$$

where $S = \langle p, m, n \rangle$.

If $S(X, Y, Z) = \sum_{i=1}^r a_i(X) b_i(Y) c_i(Z)$,
then

$$\begin{aligned} T(X, Y, Z) &= S(Z^T, X, Y^T) \\ &= \sum_{i=1}^r a_i(Z^T) b_i(X) c_i(Y^T). \end{aligned}$$

So $\text{rank}(T) \leq \text{rank}(S)$.

Same argument shows $\text{rank}(S) \leq \text{rank}(T)$,
so $\text{rank}(T) = \text{rank}(S)$. $\square$

We have two ways of combining tensors.

$$T \in \mathbb{C}^{m \times n \times p} \quad S \in \mathbb{C}^{r \times s \times t}$$

Direct sum $T \oplus S \in \mathbb{C}^{(m+r) \times (n+s) \times (p+t)}$

$$(T \oplus S)(x, y, z)$$

$$\begin{aligned} &= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p T_{ijk} x_i y_j z_k \\ &\quad + \sum_{i=m+1}^{m+r} \sum_{j=n+1}^{n+s} \sum_{k=p+1}^{p+t} S_{ijk} x_i y_j z_k \end{aligned}$$

Kronecker product : $T \otimes S \in \mathbb{C}^{mr \times ns \times pt}$

$$(T \otimes S) =$$

$$\sum_{i=1}^m \sum_{i'=1}^r \sum_{j=1}^n \sum_{j'=1}^s \sum_{k=1}^p \sum_{k'=1}^t T_{ijk} S_{i'j'k'}$$

$$\cdot X_{ii'} Y_{jj'} Z_{kk'}$$

How does rank behave under sum & product?

Lemma $R(T \oplus S) \leq R(T) + R(S)$

$$R(T \otimes S) \leq R(T) \cdot R(S).$$

Let's use this to bound w using rectangular matrix multiplication.

Lemma $\langle n, m, p \rangle \otimes \langle r, s, t \rangle$

$$= \langle nr, ms, pt \rangle.$$

Corollary If $R(\langle m, n, p \rangle) = r$,
then $\omega \leq 3 \log_{nmp} r$.

Proof

$$R(\langle m, n, p \rangle) = r$$

$$\Rightarrow R(\langle n, p, m \rangle) = R(\langle p, m, n \rangle) = r.$$

$$\begin{aligned} & \langle m, n, p \rangle \otimes \langle n, p, m \rangle \otimes \langle p, m, n \rangle \\ &= \langle nmp, nmp, nmp \rangle. \end{aligned}$$

$$R(\langle nmp, nmp, nmp \rangle) \leq r^3.$$

$$\Rightarrow \omega \leq \log_{nmp}(r^3)$$

□

We have some surprising flexibility in bounding ranks of tensors. Instead of rank, we can use border rank.

Consider the tensor corresponding to matrix mult

$$\begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$\begin{aligned} T &= X_{11} Y_{11} Z_{11} + X_{12} Y_{21} Z_{11} \\ &\quad + X_{11} Y_{12} Z_{12} + X_{12} Y_{22} Z_{12} \\ &\quad + X_{21} Y_{11} Z_{21} + X_{21} Y_{12} Z_{22}. \end{aligned}$$

Textbook algorithm $\Rightarrow R(T) \leq 6$.

But T can be approximated by a rank-5 tensor.

$$T = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(\begin{aligned} &(X_{12} + \varepsilon X_{11})(Y_{12} + \varepsilon Y_{22})Z_{12} \\ &+ (X_{21} + \varepsilon X_{11})Y_{11}(Z_{11} + \varepsilon Z_{21}) \\ &- X_{12}Y_{12}(Z_{11} + Z_{12} + \varepsilon Z_{22}) \\ &- X_{21}(Y_{11} + Y_{12} + \varepsilon Y_{21})Z_{11} \\ &+ (X_{12} + X_{21})(Y_{12} + \varepsilon Y_{21})(Z_{11} + \varepsilon Z_{22}) \end{aligned} \right)$$

We say that T has border rank 5,
written $\underline{B}(T) \leq 5$.

We will see that border rank is as good as rank, as the following lemma shows.

Lemma. If $\underline{B}(\langle n, n, n \rangle) \leq r$,
then $\omega \leq \log_n r$.