October 21, 2026

Today: Completeness of the determinant

In 1979, Valiant proved that the permanent is complete for VNP under p-projections. Today, we'll see a proof that the determinant is complete for VP under quasipolynomial-size projections. As Valiant wrote,

> "Linear algebra offers essentially the only fast technique for computing multivariate polynomials of moderate degree."

We'll prove the following theorem.

Theorem If $(f_n) \in VP$, then f_n is a projection of $\det_m$ for $m \leq n^{O(\log n)}$

Last time, we saw that if $(f_n) \in VP$, then f_n can be computed by formulas of size $n^{O(\log n)}$. So it suffices to instead prove:

Lemma If f can be computed by a formula of size s, then f is a projection of $\det_{s+1}$.

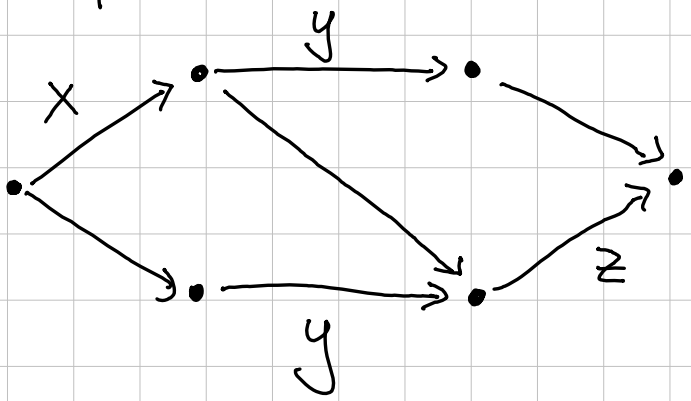
We'll prove this in two steps, using an intermediate between formulas and circuits called an algebraic branching program.

Definition An algebraic branching program is a layered directed acyclic graph G with a single source vertex s and a single sink vertex t. Edges are always between adjacent layers. Each edge is labeled by a degree-one polynomial.

If ℓ_e is the label of edge e, the branching program computes the polynomial

$$\sum_{P : s\text{-}t \text{ path}} \prod_{e \in P} \ell_e$$

For example,



computes $xy + yz + xz$.

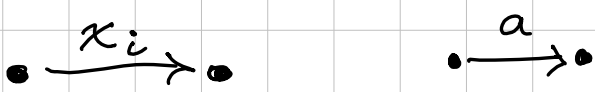
We will show

formulas of size s $\leq$ branching programs on $s+1$ vertices $\leq$ $\det_{s+1}$.

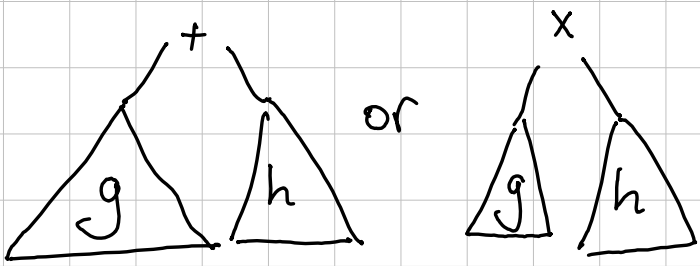
Lemma If f can be computed by a formula of size s, then it can be computed by a branching program with $s+1$ vertices where all path lengths have the same parity.

Proof. By induction on formula size.

$s = 1$: $f = x_i$ or $f = a \in \mathbb{C}$.

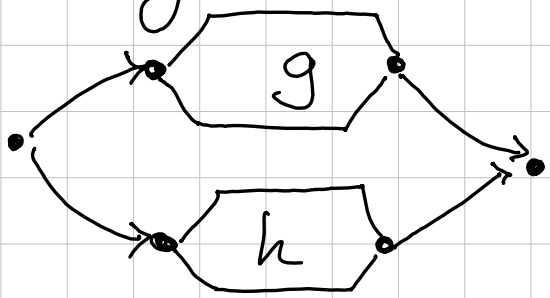


$s > 1$: $f =$



By induction, g and h have branching programs of size $s_g + 2$ and $s_h + 2$, respectively.

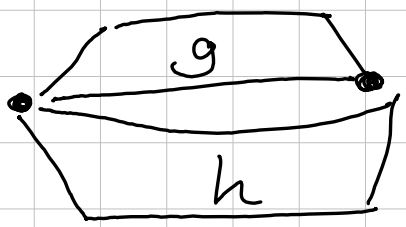
If $f = g + h$:



This ABP computes f and has size

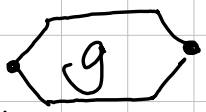
$$s_g + 1 + s_h + 1 + 2$$
$$= (s_g + s_h + 1) + 3$$
$$= s + 3.$$

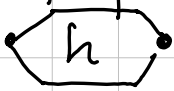
Save vertices by merging sources & merging sinks.

$f =$ 

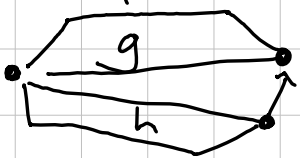
Size $(s_g + 1) + (s_h + 1) - 2$

$$= s_g + s_h = s - 1$$

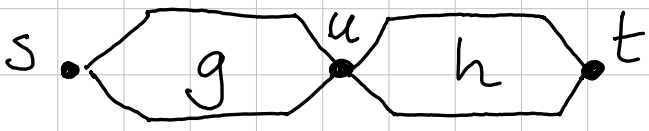
If path lengths in 

and 

have different parities, use

$f =$ 

If $f = g \times h$:



This ABP computes f :

$$\sum_{\substack{P: s \to t \\ \text{path}}} \prod_{e \in P} \ell_e$$

$$= \sum_{\substack{P_1: s \to u \\ \text{path}}} \sum_{\substack{P_2: u \to t \\ \text{path}}} \left(\prod_{e \in P_1} \ell_e \right) \left(\prod_{e \in P_2} \ell_e \right)$$

$$= \left(\sum_{P_1: s \to u} \prod_{e \in P_1} \ell_e \right) \left(\sum_{P: u \to t} \prod_{e \in P} \ell_e \right)$$

$$= g \times h.$$

The ABP has size

$$(s_g + 1) + (s_h + 1) - 1 = s_f$$

□

Now we convert ABPs to determinants

Lemma. If f can be computed by an ABP of size s,* then f is a (linear) projection of the $s \times s$ determinant.

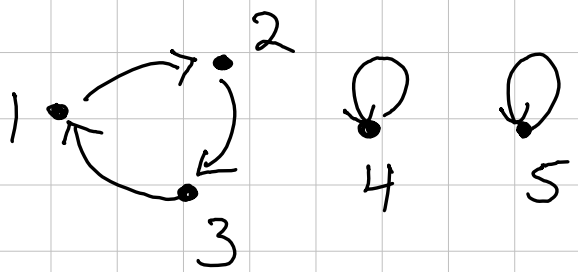
*where all path lengths have same parity...

Proof Let $A = (a_{i,j})$ be the adjacency matrix of the ABP. Recall that

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)}.$$

Every monomial in $\det(A)$ corresponds to a cycle cover in G: a set of vertex-disjoint cycles that cover all vertices in G.

Example: the permutation $(1\ 2\ 3)(4)(5)$ corresponds to



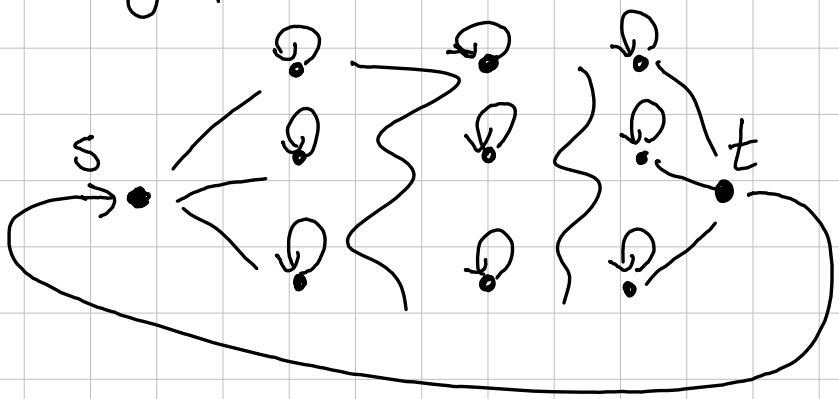
Fact: $\operatorname{sgn}(\sigma) = (-1)^{\#\text{even cycles in } \sigma}$

So
- ABPs sum over paths
- det sums over cycle covers.

To turn paths into cycle covers, add

- self-loops to all vtxs except s and t, of weight 1
- edge of wgt ± 1 from t to s.

Now graph looks like



Obs ∃ one-to-one correspondence btwn cycle covers & paths in original graph.

All path lengths have same parity
$\Rightarrow$ all cycle covers have same sign.

If A' = adj matrix of new graph,
then

$$\det(A') = \pm \sum_{\substack{\text{cycle} \\ \text{cover } C}} \prod_{e \in C} l_e$$

$$= \sum_{P : s \to t \text{ path}} \prod_{e \in P} l_e$$

$$= f.$$

□