October 23, 2026

Admin

- Robert out of town next week. Asynchronous lectures will be posted to Canvas
- HW 3 & project proposals due next Wednesday
- Office hours by appointment only next week.

Today: VNP-completeness of the permanent.

Recall that the permanent of an $n \times n$ matrix is given by

$$\operatorname{perm}(X) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i,\sigma(i)},$$

where S_n is the group of permutations on $\{1, 2, \ldots, n\}$.

For example,

$$\operatorname{perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc$$

$$\operatorname{perm}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh + ceg + bdi + afh.$$

We'll prove that the p-family (perm_n) is VNP-complete. That is,

- $(\operatorname{perm}_n) \in \mathrm{VNP}$
- For every $(f_n) \in \mathrm{VNP}$, (f_n) is a p-projection of (perm_n).

<u>Lemma.</u> $(\operatorname{perm}_n) \in \mathrm{VNP}$

<u>Proof.</u> Recall this means we need to show

$$\operatorname{perm}_n(X) = \sum_{e \in \{0,1\}^m} g_{\mathrm{poly}(n)}(X, e)$$

for some $(g_n) \in \mathrm{VP}$.

Idea: Sum over matrices $E \in \{0,1\}^{n \times n}$.
Design g so that

$$g(X,E) = \begin{cases} \prod_{i=1}^{n} X_{i,\sigma(i)} & \text{if } E \text{ is permutation matrix of } \sigma \\ 0 & \text{otherwise.} \end{cases}$$

If we do this, then

$$\sum_{E \in \{0,1\}^{n \times n}} g(X,E) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i,\sigma(i)}$$
$$= \text{perm}(X)$$

as desired.

When is a $\{0,1\}$-matrix E a permutation matrix?

- Exactly one 1 in every row, and
- Exactly one 1 in every column

$$\alpha(Y) = \prod_{i,j,k=1}^{n} (1 - Y_{i,j} Y_{i,k})$$

Claim $\alpha(E) = \begin{cases} 1 & \text{every row of } E \text{ has} \leq \text{one } 1 \\ 0 & \text{some row has} \geq \text{two ones} \end{cases}$

$$\beta(Y) = \prod_{i=1}^{n} \sum_{j=1}^{n} Y_{i,j}$$

Claim If $\alpha(E) = 1$, then

$\beta(E) = \begin{cases} 1 & \text{every row of } E \text{ has one } 1 \\ 0 & \text{some row of } E \text{ is all zeroes} \end{cases}$

Claim

$$\alpha(E)\beta(E)\alpha(E^T)\beta(E^T) = \begin{cases} 1 & E \text{ permutation matrix} \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma(X,Y) = \prod_{i=1}^{n} \sum_{j=1}^{n} X_{i,j}\, Y_{i,j}$$

Claim If $E \in \{0,1\}^{n \times n}$ is a permutation matrix corresponding to permutation σ, then

$$\gamma(X,E) = \prod_{i=1}^{n} X_{i,\sigma(i)}.$$

Let

$$g(X,Y) = \alpha(Y)\,\beta(Y)\,\alpha(Y^T)\,\beta(Y^T) \cdot \gamma(X,Y)$$

Claim: $g \in VP$

Proof $\deg(g) \leq O(n^3)$.
Definitions of α, β, γ give circuits for g. □

Claim $\sum_{E \in \{0,1\}^{n \times n}} g(X,E) = \mathrm{perm}(X)$.

Proof If E not permutation matrix, then $\alpha(E)\beta(E)\alpha(E^T)\beta(E^T) = 0$, so $g(X, E) = 0$. Thus

$$\sum_{E \in \{0,1\}^{n \times n}} g(X, E)$$

$$= \sum_{\substack{E \in \{0,1\}^{n \times n} \\ \text{permutation}}} \alpha(E)\beta(E)\alpha(E^T)\beta(E^T)\gamma(X, E)$$

$$= \sum_{\substack{E \in \{0,1\}^{n \times n} \\ \text{permutation}}} \gamma(X, E)$$

$$= \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i,\sigma(i)} = \text{perm}(X) \quad \square$$

There is a more general result along these lines called **Valiant's criterion**

Theorem If $\varphi : \{0,1\}^* \to \mathbb{N}$ computable in poly time, then

$$f_n = \sum_{e \in \{0,1\}^n} \varphi(e) x_1^{e_1} \cdots x_n^{e_n}$$

is in VNP.

Next, we'll work towards a proof that (perm_n) is VNP-hard. We need one structural result about VNP without proof.

Definition

- $\mathrm{VF} = \left\{ \begin{array}{l} \text{p-families computable} \\ \text{by poly-size formulas} \end{array} \right\}$
- VNP_e : a p-family (f_n) is in VNP_e if $\exists\, (g_n) \in \mathrm{VF}$ such that

$$f_n(x) = \sum_{e \in \{0,1\}^m} g_{n+m}(x, e)$$

for $m \leq \mathrm{poly}(n)$.

Theorem $\mathrm{VNP} = \mathrm{VNP}_e$.

Theorem (perm_n) is VNP-hard.

Proof let $(f_n) \in \text{VNP} = \text{VNP}_e$.
We need to show (f_n) is a p-projection of (perm_n).

Let $(g_n) \in \text{VF}$ such that $f(x) = \sum_e g(x,e)$.
Last time, we saw how to construct a weighted graph G st

$$g(x,y) = \sum_{\substack{C : \text{cycle} \\ \text{cover in } G}} \prod_{e \in C} l_e .$$

We will modify this graph to implement the sum over $e \in \{0,1\}^m$.

As a warm-up, suppose y_i labels only a single edge in G. Then we can write

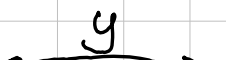
$$g(x,y) = y_i \cdot g_1 + g_0$$

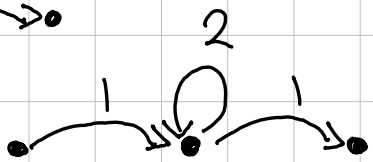
Summing over $e_1 \in \{0, 1\}$ gives us the polynomial

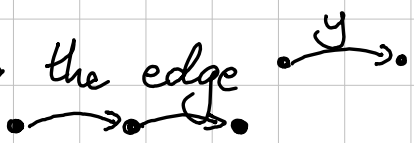
$$g_1 + 2g_0.$$

So we want to

- double weight of all cycle covers that don't use the edge labeled by y
- keep weight of cycle covers using the edge labeled by y the same (after dividing by y)

Replace 

with the gadget 

- Cycle covers using the edge y are extended using 
- Cycle covers not using y are extended using 

This allows us to sum over $\{0, 1\}$ assignments to one edge.

The difficulty arises when many edges share the same variable as a label.
For each y variable, we want

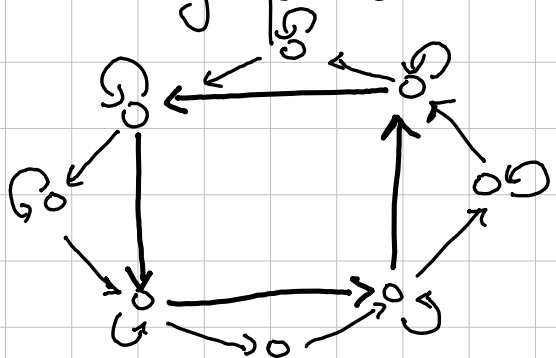
- every cycle cover using at least one y edge should have the same weight
- cycle covers using no y edges have their weight doubled.

We'll accomplish this using two gadgets.

Gadget 1: the r-rosette.

An r-rosette is a graph on $2r$ vertices.

4-rosette:

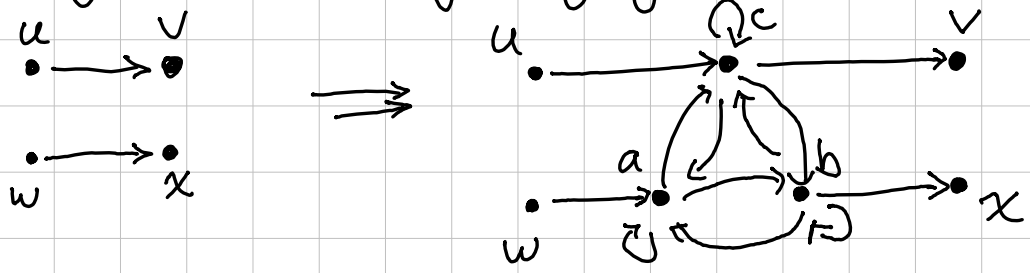


Call the bolded edges **connector edges**.

Obs An r-rosette has

- $2^r - 1$ cycle covers using connector edges, one for every nonempty subset of connector edges
- 2 cycle covers using no connector edges

Gadget 2: the glue gadget



The graph on $\{a, b, c\}$ has adjacency matrix

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} -1 & 1 & 1/2 \\ 1 & 1 & -1/2 \\ 1 & 1 & -1/2 \end{bmatrix} \end{array}$$

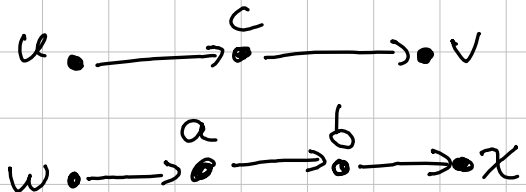
Claim Let (u,v) and (w,x) be two edges in G. Let G' be graph obtained by replacing these edges w/ glue gadget. Then

$$\text{perm}(G') = \sum_{C} \prod_{e \in C} l_e ,$$

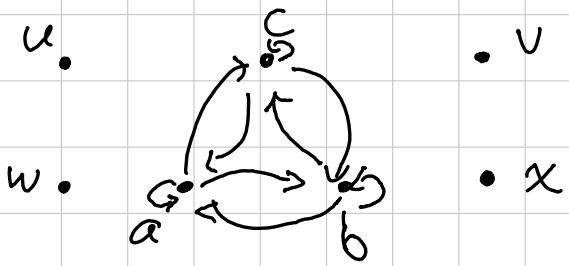
where the sum is over cycle covers of G that either use (a) both (u,v) and (w,x), or (b) use neither edge.

Proof:

Use both :



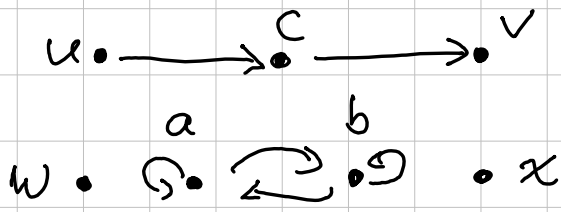
Use neither :



Check that

$$\text{perm}\begin{pmatrix} -1 & 1 & 1/2 \\ 1 & 1 & -1/2 \\ 1 & 1 & -1/2 \end{pmatrix} = 1 .$$

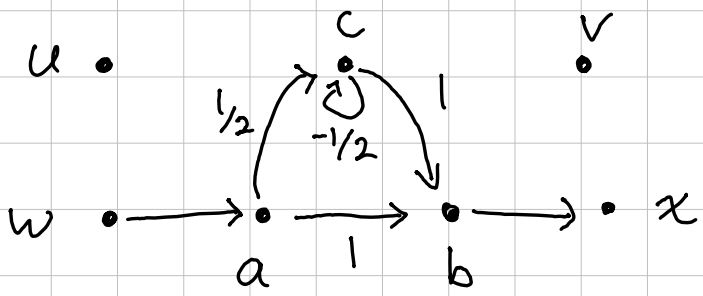
Use (u, v) but not (w, x):



$$\operatorname{perm}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = 0$$

So net contribution is zero.

Use (w, x) but not (u, v):



Net contribution again zero. □

Now back to completeness of the permanent.

Given graph G, modify it to get G' as follows:

For each y variable y_i:
$r_i \leftarrow$ # edges labeled y_i
Add an r_i-rosette
Glue each connector edge to one of the y_i edges.

If $\text{perm}(G) = g(x, y)$,

then $\text{perm}(G') = \sum_{e \in \{0,1\}^m} g(x, e)$

Proof by induction on y variables. By construction of the gadgets, the polynomial $g_1 y + g_0$ becomes $g_1 + 2g_0$. So one step correctly sums over $\{0, 1\}$ assignments to one variable. □