Till now we’ve seen signals that do not change in frequency over time. How do we modify the signal to obtain a time-varying frequency?

A chirp signal is one that sweeps linearly from a low to a high frequency.

Can we create such a signal by concatenating small sequences, each with a frequency that is higher than the last?

This approach will likely lead to problems lining up the phase of each segment so that discontinuities aren’t introduced in the resulting waveform (as seen below).

A better approach is to modify the equation for the sinusoid so that the frequency is time-varying.

Recall that the original equation for a sinusoid is given by

\[ x(t) = A \cos(\omega_0 t + \phi) \]

where the instantaneous phase, given by \((\omega_0 t + \phi)\), changes linearly with time.

Notice that the time derivative of the phase is the radian frequency of the sinusoid \(\omega_0\), which in this case is a constant.

More generally, if

\[ x(t) = A \cos(\theta(t)) \]

the instantaneous frequency is given by

\[ \omega(t) = \frac{d}{dt}\theta(t). \]

Now, let’s make the phase quadratic, and thus non-linear, rather than linear with respect to time.

\[ \theta(t) = 2\pi \mu t^2 + 2\pi f_0 t + \phi. \]

The instantaneous radian frequency, which is the derivative of the phase \(\theta\), now becomes

\[ \omega_i(t) = \frac{d}{dt}\theta(t) = 4\pi \mu t + 2\pi f_0 \]

which in Hz is

\[ f_i(t) = 2\mu t + f_0. \]

Notice the frequency is no longer a constant but is changing linearly in time, where \(f_0\) is the starting frequency and

\[ k = 2\mu = \frac{f_1 - f_0}{T} \]

is the rate of frequency increase over duration \(T\).

The sinusoid with frequency sweeping from \(f_0\) to \(f_1\) (i.e. with instantaneous phase having a time derivative equal to \(2\pi(k t + f_0)\)) is given by:

\[ \cos(2\pi \frac{k}{2} t^2 + 2\pi f_0 t) \]
Sweeping Frequency

- We can use this time-varying value for the instantaneous frequency in our original equation for a sinusoid,

\[ x(t) = A \cos(2\pi f(t) t + \phi), \]

where

\[ f(t) = \frac{k}{2} t + f_0. \]

Vibrato simulation

- Vibrato is a term used to describe a wavering of pitch.
- Vibrato occurs very naturally in the singing voice (though some may say a little exaggerated in some operatic performances), and in instruments where the musician has control after the note has been played (such as the violin, wind instruments, the theremin, etc.).
- In Vibrato, the frequency does not change linearly (as our last chirp signal example) but rather sinusoidally, creating a sense of a wavering pitch.
- Since the instantaneous frequency of the sinusoid is the derivative of the instantaneous phase, and the derivative of a sinusoid is a sinusoid, we merely apply a sinusoidal signal to the instantaneous phase of a carrier signal to create vibrato:

\[ x(t) = A_c \cos(2\pi f_c t + A_m \cos(2\pi f_m t + \phi_m) + \phi_c) \]

Vibrato cont.

- We can therefore use FM synthesis to create a vibrato effect, where the instantaneous frequency of the carrier oscillator varies over time according to the parameters controlling
  - the width of the vibrato (the deviation from the carrier frequency)
  - the rate of the vibrato.
- The width of the vibrato is determined by the amplitude of the modulating signal, \( A_m \).
- The rate of vibrato is determined by the frequency of the modulating signal, \( f_m \).
- In order for the effect to be perceived as vibrato, the vibrato rate must be below the audible frequency range and the width made quite small.
FM Synthesis of Musical Instruments

• When the vibrato rate is in the audio frequency range, and when the width is made larger, the technique can be used to create a broad range of distinctive timbres.

• Frequency modulation (FM) synthesis was invented by John Chowning at Stanford University’s Center for Computer Research in Music and Acoustics (CCRMA).

• FM synthesis uses fewer oscillators than either additive or AM synthesis to introduce more frequency components in the spectrum.

• Where AM synthesis uses a signal to modulate the amplitude of a carrier oscillator, FM synthesis uses a signal to modulate the frequency of a carrier oscillator.

Frequency Modulation

• The general equation for an FM sound synthesizer is given by

\[ x(t) = A(t) \cos(2\pi f_c t + I(t) \cos(2\pi f_m t + \phi_m) + \phi_c) , \]

where

- \( A(t) \) \( \triangleq \) the time varying amplitude
- \( f_c \) \( \triangleq \) the carrier frequency
- \( I(t) \) \( \triangleq \) the modulation index
- \( f_m \) \( \triangleq \) the modulating frequency
- \( \phi_m, \phi_c \) \( \triangleq \) arbitrary phase constants.

Modulation Index

• The function \( I(t) \), called the modulation index envelope, determines significantly the harmonic content of the sound.

• To see this, let’s first look at how it affects the instantaneous frequency \( f_i(t) \).

• Given the general FM equation (with unit amplitude and phases set to zero for simplicity)

\[ x(t) = \cos(2\pi f_c t + I(t) \cos(2\pi f_m t)) , \]

the instantaneous frequency \( f_i(t) \) (in Hz) is given by

\[ f_i(t) = \frac{1}{2\pi} \frac{d}{dt} \theta(t) \]

\[ = \frac{1}{2\pi} \frac{d}{dt} [2\pi f_c t + I(t) \cos(2\pi f_m t)] . \]

Modulation Index and Instantaneous Frequency

• It is not necessary to completely solve for \( f_i(t) \) to get a sense for how it, and the harmonic content, is affected by the modulation index.

• The result on the previous slide can be further simplified by assuming \( I(t) \) does not change over time:

\[ f_i(t) = \frac{1}{2\pi} \frac{d}{dt} [2\pi f_c t + I \cos(2\pi f_m t)] . \]

• Applying the product and chain rules yield

\[ f_i(t) = \frac{1}{2\pi} [2\pi f_c - I \sin(2\pi f_m t) 2\pi f_m] \]

\[ = f_c - I \sin(2\pi f_m t) f_m . \]

• The second term consists of a sinusoidal variation of frequency \( f_m \), with amplitude \( I(t) f_m \).

• This indicates that the value of \( I \) determines the maximum amount by which the instantaneous frequency deviates from the carrier frequency \( f_c \).
Changing harmonic content over time

- Since $I(t)$ is actually a function of time, this deviation, seen as sidebands at multiples of $f_m$ in the synthesized sound, may vary with time.
- The fact that $I(t)$ is time varying, allows us to change the timbre of a sound over time, a known characteristic of musical sounds.

**FM Sidebands**

- The upper and lower sidebands produced by FM are grouped in pairs according to the harmonic number of $f_m$, that is, frequencies present are given by $f_c \pm kf_m$.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_c - f_m$</td>
<td>-0.5</td>
</tr>
<tr>
<td>$f_c - 3f_m$</td>
<td>0.5</td>
</tr>
<tr>
<td>$f_c + f_m$</td>
<td>-0.5</td>
</tr>
<tr>
<td>$f_c + 3f_m$</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Figure 3: Sidebands produced by FM synthesis.

**Determining the Modulation index**

- Though it is possible to determine the exact amplitude of the sidebands (see next slide), we can also make use of a rule of thumb.
- In *Computer Music*, the modulation index $I$ is given by

$$I = \frac{d}{f_m}$$

where $d$ is the amount of frequency deviation produced by the modulating oscillator.
- When $d = 0$, the index $I$ is also zero, and no modulation occurs. Increasing $d$ causes the sidebands to acquire more power at the expense of the power in the carrier frequency.
- The deviation $d$ can therefore act as a control on FM bandwidth.

**Bessel Functions of the First Kind**

- The amplitude of the $k^{th}$ sideband is given by $J_k(I)$, where $J_k$ is a Bessel function of the first kind, of order $k$.

<table>
<thead>
<tr>
<th>$J_0$</th>
<th>$J_1$</th>
<th>$J_2$</th>
<th>$J_3$</th>
<th>$J_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5</td>
<td>0.5</td>
<td>-0.5</td>
<td>0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.5</td>
<td>0.5</td>
<td>-0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Figure 4: Bessel functions of the first kind, plotted for orders 0 through 5.

- From this plot we see that higher order Bessel functions, and thus higher order sidebands, do not have significant amplitude when $I$ is small.

*Bessel functions are solutions to Bessel’s differential equation.*
• Higher values of $I$ therefore produce higher order sidebands.
• In general, the highest-ordered sideband that has significant amplitude is given by the approximate expression $k = I + 1$.
• To produce a Bessel function $J_k(I)$ in Matlab:
  \[ \text{besselj}(k, I); \]
• Notice for there to be significant energy in the 5th sideband, $I$ must be at least 4:
  \[
  \begin{align*}
  \gg & \text{besselj}(5, 2) \\
  \text{ans} &= \\
  0.0070 \\
  \gg & \text{besselj}(5, 3) \\
  \text{ans} &= \\
  0.0430 \\
  \gg & \text{besselj}(5, 4) \\
  \text{ans} &= \\
  0.1321
  \end{align*}
  \]

**Odd-Numbered Lower Sidebands**

• The amplitude of the odd-numbered lower sidebands is the appropriate Bessel function multiplied by -1, since odd-ordered Bessel functions are odd functions. That is
  \[ J_{-k}(I) = -J_k(I). \]

**Effect of Phase in FM**

• The phase of a spectral component does not have an audible effect unless other spectral components of the same frequency are present.
• In the case of frequency overlap, the amplitudes will either add or subtract and the tone of the sound will change as a result.
• If the FM spectrum contains frequency components below 0 Hz, they are folded over the 0 Hz axis to their corresponding positive frequencies.
• In the case that the carrier is an odd function (such as a sine instead of a cosine), the act of folding reverses the phase. A sideband with a negative frequency is equivalent to a component with the corresponding positive frequency but with the opposite phase.

**FM Spectrum**

The frequencies present in a simple FM spectrum are $f_c \pm kf_m$, where $k$ is an integer greater than zero. The carrier frequency component is at $k = 0$.

**Figure 5:** Bessel functions of the first kind, plotted for odd order s.

**Figure 6:** Spectrum of a simple FM instrument, where $f_c = 220$, $f_m = 110$, and $I = 2$.

**Figure 7:** Spectrum of a simple FM instrument, where $f_c = 900$, $f_m = 600$, and $I = 2$. 
Fundamental Frequency in FM

• In determining the fundamental frequency of your FM sound, it is useful to represent the ratio of the carrier and modulator frequencies as a reduced fraction, that is:
  \[ \frac{f_c}{f_m} = \frac{N_1}{N_2} \]

  where \( N_1 \) and \( N_2 \) are integers with no common factors.

• The fundamental frequency is then given by
  \[ f_0 = \frac{f_c}{N_1} = \frac{f_m}{N_2}. \]

• As in the previous plot for example, a carrier frequency \( f_c = 220 \) and modulator frequency \( f_m = 110 \) yields the ratio of
  \[ \frac{f_c}{f_m} = \frac{220}{110} = \frac{2}{1} = \frac{N_1}{N_2}. \]

  and a fundamental frequency of
  \[ f_0 = \frac{220}{2} = \frac{110}{1} = 110. \]

Missing Harmonics in FM

• If \( N_2 = M \) where \( M \) is an integer greater than 1, then every \( M^{th} \) harmonic of \( f_0 \) is missing in the spectrum.

• This can be seen in the plot below where the ratio of the carrier to the modulator is 4:3 and \( N_2 = 3 \). Notice the fundamental frequency \( f_0 \) is 100, but every third multiple of \( f_0 \) is missing from the spectrum.

Some FM instrument examples

• When implementing simple FM instruments, we have several basic parameters that will effect the overall sound:
  1. The duration,
  2. The carrier and modulating frequencies
  3. The maximum (and in some cases minimum) modulating index scalar
  4. The envelopes that define how the amplitude and modulating index evolve over time.

• Using the information taken from John Chowning’s article on FM (details of which appear in the text *Computer Music* (pp. 125-127)), we may develop envelopes for the following simple FM instruments:
  - bell-like tones,
  - wood-drum
  - brass-like tones
  - clarinet-like tones
Formants

• Another characteristic of sound, in addition to its spectrum, is the presence of formants.
• The formants describe certain regions in the spectrum where there are strong resonances (where the amplitude of the spectral components is considerably higher).
• We may view formants as the peaks in the spectral envelope.
• As an example, pronounce aloud the vowels “a”, “e”, “i”, “o”, “u” while keeping the same pitch for each. Since the pitch is the same, we know the integer relationship of the spectral components is the same.
• The formants are what allows us to hear a difference between the vowel sounds.

Two Carrier Oscillators

• In FM synthesis, the peaks in the spectral envelop can be controlled using an additional carrier oscillator.
• In the case of a single oscillator, the spectrum is centered around a carrier frequency. With an additional oscillator, an additional spectrum may be generated that is centered around a formant frequency.
• When the two signals are added, their spectra are combined.
• If the same oscillator is used to modulate both carriers (though likely using separate modulation indeces), and the formant frequency is an integer multiple of the fundamental, the spectra of both carriers will combine in such a way the the components will overlap, and a peak will be created at the formant frequency.
Two Carriers cont.

- In Figure 10, both carriers are modulated by the same oscillator with a frequency \( f_m \).
- The index of modulation for the first and second carrier is given by \( I_1 \) and \( I_2/I_1 \) respectively.
- The value \( I_2 \) is usually less than \( I_1 \), so that the ratio \( I_2/I_1 \) is small and the spectrum does not spread too far beyond the region of the formant.
- The frequency of the second carrier \( f_{c2} \) is chosen to be a harmonic of the fundamental frequency \( f_0 \) that it is close to the desired formant frequency \( f_f \) (from Computer Music). That is
  \[
  f_{c2} = n f_0 = \text{int}(f_f/f_0 + 0.5)f_0.
  \]
  
- This ensures that the second carrier frequency remains harmonically related to \( f_0 \). If \( f_0 \) changes, the second carrier frequency will remain as close as possible to the desired formant frequency \( f_f \) while remaining an integer multiple of the fundamental frequency \( f_0 \).