Sinusoids

- Sinusoids are fundamental in physics/acoustics and simple harmonic motion.
- A sinusoid is a function of time having the following form:
  \[ x(t) = A \sin(\omega t + \phi), \]
  where \( x \) is the quantity which varies over time and
  \[ A \triangleq \text{peak amplitude} \]
  \[ \omega \triangleq \text{radian frequency (rad/sec)} = 2\pi f \]
  \[ f \triangleq \text{frequency (Hz)} \]
  \[ t \triangleq \text{time (seconds)} \]
  \[ \phi \triangleq \text{initial phase (radians)} \]
  \[ \omega t + \phi \triangleq \text{instantaneous phase (radians)} \]

Amplitude and Magnitude.

- The term **peak amplitude**, often shortened to **amplitude**, is the nonnegative value of the waveform’s peak (either positive or negative).
- The **instantaneous amplitude** of \( x \) is the value of \( x(t) \) (either positive or negative) at time \( t \).
- The **instantaneous magnitude**, or simply **magnitude**, of \( x \) is nonnegative and is given by \(|x(t)|\).

Phase and Period.

- The **initial phase** \( \phi \), given in radians, tells us the position of the waveform cycle at \( t = 0 \). Also sometimes called:
  - phase offset
  - phase shift
  - phase factor
- One **cycle** of a sinusoid is \( 2\pi \) radians.
- The **period** \( T \) of a sinusoid is the time (in seconds) it takes to complete one cycle.
- Since sinusoids are periodic with period \( 2\pi \), an initial phase of \( \phi \) is indistinguishable from an initial phase of \( \phi \pm 2\pi \). We may therefore restrict the range of \( \phi \) so that it does not exceed \( 2\pi \).
Frequency

- The frequency \( f \) of the waveform is given in cycles per second or Hertz (Hz).
- Frequency is equivalent to the inverse of the period \( T \) of the waveform,
  \[ f = \frac{1}{T} \text{ Hz}. \]
- The radian frequency \( \omega \), given in radians per second, is equivalent to the frequency in Hertz scaled by \( 2\pi \),
  \[ \omega = 2\pi f \text{ (rad/sec)}. \]

Sine and cosine functions.

- The sine and cosine function are very closely related and can be made equivalent simply by adjusting their initial phase:
  \[ \sin \theta = \cos \left( \theta - \frac{\pi}{2} \right) \quad \text{or} \quad \cos \theta = \sin \left( \theta + \frac{\pi}{2} \right). \]

Time-shifting a signal.

- If a signal can be expressed in the form \( x(t) = s(t - t_1) \), we say \( x(t) \) is a **time-shifted** version of \( s(t) \).

\[
x(t) = s(t-2) = s(t-2) = t - 2 \quad 0 \leq t - 2 \leq 1 \\
x(t) = s(t+1) = s(t+1) = t + 1 \quad -1 \leq t + 1 \leq 2
\]

- \( x(t) \) is simply the \( s(t) \) function with its origin shifted to the right, or **delayed**, by 2 seconds.
- Similarly, \( y(t) \) is equivalent to \( s(t) \) with its origin shifted to the left, or **advanced in time** by 1 second.
- A positive phase indicates a shift to the left whereas a negative phase indicates a shift to the right.

Sinusoids and physical motion

Stricking a tuning fork causes its tines to vibrate, moving back and forth through an equilibrium position, producing a tone that is almost sinusoidal.

The relationship of sinusoids to physical acoustic vibration can be shown using several important law's of physics:

- **Newton's second law**, given by \( F = ma \): The tine overshoots the equilibrium position due to the force caused by the tine's mass \( m \) and acceleration \( a \).
- **Hooke's law**, given by \( F = kx \): When the tine overshoots the equilibrium position, there is a restoring force, proportional to the tine’s displacement \( x \) and the tine’s stiffness \( k \), that brings it back toward equilibrium. That is,
  \[ ma = -kx. \]
Displacement, Velocity and Acceleration

- **Velocity**, \( v \), is the rate at which the displacement, \( x \), changes. It is given by the derivative of displacement with respect to time:

  \[
  v = \frac{dx}{dt}.
  \]

- **Acceleration**, \( a \), is the rate at which velocity changes. It is given by the derivative of velocity with respect to time, or alternatively, the second derivative of displacement with respect to time. That is,

  \[
  a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.
  \]

- **Displacement** \( x \) is, therefore, a function proportional to its first and second derivative.

Simple Harmonic Motion

- Recall that a sinusoid is a function proportional to its derivative:

  \[
  \frac{d}{dt} \sin(\omega t) = \omega \cos(\omega t) \]

  \[
  \frac{d}{dt} \cos(\omega t) = -\omega \sin(\omega t),
  \]

  where \( \omega \) is the radian frequency of the sine and cosine functions.

- Sinusoids are also proportional to their second derivatives:

  \[
  \frac{d^2}{dt^2} \sin(\omega t) = -\omega^2 \sin(\omega t) \]

  \[
  \frac{d^2}{dt^2} \cos(\omega t) = -\omega^2 \cos(\omega t).
  \]

Simple Harmonic Motion cont.

- The sinusoidal motion of the tuning fork is called **simple harmonic motion**, which is the simplest form of motion in vibrating systems.

- If the displacement of a vibrating system is sinusoidal, i.e. \( x = \cos(\omega t) \), we may use Newton’s third law to determine the frequency of the system. That is:

  \[
  ma = -kx
  \]

  \[
  -\omega^2 \cos(\omega t) = -\frac{k}{m} \cos(\omega t)
  \]

  \[
  \omega = \sqrt{\frac{k}{m}}
  \]

  That is, the frequency is proportional to the square root of the ratio of the stiffness (or spring constant), \( k \) (given in Newtons per meter), to the mass, \( m \).

Sinusoidal and Circular Motion

- Consider a vector of length one (1), rotating at a steady speed in a plane, the vector tracing a circle with a radius equal to its length.

  Figure 4: A vector rotating along the unit circle.

- Each time the vector completes one rotation of the circle, it has completed a cycle of \( 2\pi \).

- The rate at which the vector completes one cycle is given by its frequency.

- The length of the vector is given by its amplitude (which for simplicity, in this case, is one (1)).
Sinusoidal and Circular Motion cont.

- The x- and y-axis are the horizontal and vertical lines intersecting at the circle’s centre.

![Diagram](image)

Figure 5: The vector coordinates are determined by projecting onto the x and y-axis.

- Projecting the vector onto the x- and y-axes allows us to determine its coordinates in the xy-plane.
- If the vector is rotated in a counterclockwise direction, at angle $\theta$ from the positive x-axis, projecting onto both the x- and y-axes creates right angle triangles.
- Trigonometric identities, with knowledge of $\theta$ and the vector length, will help us determine the coordinates.

Adding two sinusoids of the same frequency

- Adding two sinusoids of the same frequency but with possibly different amplitudes and phases, produces another sinusoid with that frequency.

![Graph](image)

Figure 7: Adding two sinusoids of the same frequency.

- Recalling the expression for sinusoidal motion and trigonometric identities, we may see that

$$x = A \cos(\omega_0 t + \phi)$$
$$= A \cos(\phi + \omega_0 t)$$
$$= [A \cos \phi] \cos \omega_0 t + [-A \sin \phi] \sin \omega_0 t$$
$$= B \cos \omega_0 t + C \sin \omega_0 t$$

where the amplitude $A$ is given by

$$A = \sqrt{B^2 + C^2},$$

and the phase $\phi$ is given by

$$\phi = \tan^{-1}\left(\frac{C}{B}\right).$$

Sinusoids and Circular Motion cont.

- Projecting onto the x- and y-axis gives a sequence of points that resemble a cosine and sine function respectively.

![Graph](image)

Figure 6: Projecting onto the x and y-axis.
Vector Addition

- Since one vector represents one sinusoid, to add two sinusoids of the same frequency, we need only perform vector addition.

![Figure 8: Adding sinusoids using vector addition.]

- Since the vectors have the same frequency, they will rotate as a unit and their sum will have the same frequency.
- The sum vector $u + v$ in Figure 8 also has its own $x$ and $y$ component (from projecting onto the $x$- and $y$-axes) and therefore may have a different amplitude and phase.

Complex numbers

- Complex numbers provides a system for manipulating rotating vectors, and allows us to represent the geometric effects of common digital signal processing operations, like filtering, in algebraic form.
- In rectangular (or Cartesian) form, the complex number $z$ is defined by the notation
  \[ z = x + jy. \]
- The part without the $j$ is called the real part, and the part with the $j$ is called the imaginary part.

Exponentials

- The exponential is typically used to describe the natural growth or decay of a system’s state.
- An exponential is defined as
  \[ x(t) = e^{-t/\tau} \]
  where $e = 2.7182...$ and $\tau$ is the characteristic time constant of the exponential where at time $t = \tau$, the function will equal $1/e$.
- Though this function may, at first, seem very different from a sinusoid, both functions are aspects of a slightly more complicated function.

Complex Numbers cont.

- A complex number can be drawn as a vector, where the tip of the vector is at the point $(x, y)$, where $x$ is the horizontal coordinate, or the real part, and $y$ is the vertical coordinate, or the imaginary part.
- We may now refer to the $x$ and $y$ axes as the real- and imaginary-axes, respectively.
- A multiplication by $j$ may be seen as an operation meaning “rotate counterclockwise 90° or $\pi/2$”.
- Two successive rotations by $\pi/2$ bring us to the negative real axis, that is, $j^2 = -1$. From this we see $j = \sqrt{-1}$. 
Polar Form

- A complex number may also be represented in polar form
  \[ z = re^{j\theta}. \]
  where the vector is defined by its
  1. length \( r \)
  2. direction \( \theta \)
- The length of the vector is also called the magnitude of \( z \) (denoted \( |z| \)) and the angle with the real axis is called the argument of \( z \) (denoted \( \arg z \)).
- Using trigonometric identities and the Pythagorean theorem, we can compute:
  1. The Cartesian coordinates \((x, y)\) from the polar variables \( r \angle \theta \):
     \[ x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \]
  2. The polar coordinates from the Cartesian:
     \[ r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan \left( \frac{y}{x} \right) \]

Euler’s Formula

- Recall from our previous section on sinusoids that the projection of a rotating sinusoid on the \( x- \) and \( y- \) axes, traces out a cosine and a sine function respectively.
- From this we can see how Euler’s famous formula for the complex exponential was obtained by substituting for \( x \) and \( y \) as follows:
  \[ e^{j\theta} = \cos \theta + j \sin \theta \] (2)
  Equation 2 is valid for any point \((\cos \theta, \sin \theta)\) on a circle of radius 1. This can be further generalized to be valid for any complex number \( z \)
  \[ z = re^{j\theta} = r \cos \theta + jr \sin \theta; \] (3)
- Though called “complex”, these numbers usually simplify calculations considerably—particularly in the case of multiplication and division.