Knaster-Tarski theorem:

A monotone operator on a complete lattice has a complete lattice of fixpoints; thus, the least and the greatest fixpoint.

Obtained by Tarski in 1936 (or 1939), published in a joint paper with Knaster in 1955.

Monotone Functions

<u>Def.</u> Let S be a set (of states) and $F : \mathcal{P}ow(S) \to \mathcal{P}ow(S)$
(F maps sets to sets)

- F is <u>monotone</u> if

sets of states

$$X \subseteq Y \Rightarrow F(X) \subseteq F(Y)$$

sets of states

- X is a <u>fixpoint</u> of F, if

$$F(X) = X$$

<u>Ex.1.</u> $S \stackrel{def}{=} \{s_0, s_1\}$

set set

$F(Y) \stackrel{def}{=} Y \cup \{s_0\}$ for all $Y \subseteq S$

~~F(...)~~ F is monotone:

$$Y \subseteq Y' \Rightarrow Y \cup \{s_0\} \subseteq Y' \cup \{s_0\}$$

Ex 2. $G(Y) \stackrel{def}{=}$ if $Y = \{s_0\}$ then $\{s_1\}$
else $\{s_0\}$

$$\begin{cases} \{s_0\} \stackrel{G}{\longmapsto} \{s_1\} \\ \mathcal{P}ow(S) \setminus \{s_0\} \longmapsto \{s_0\} \end{cases}$$

Not monotone:

$$\begin{array}{ccc} \{s_0\} & \subseteq & \{s_0, s_1\} \\ G \downarrow & & \downarrow G \end{array}$$

$$G(\{s_0\}) = \{s_1\} \not\subseteq \{s_0\} = G(\{s_0, s_1\})$$

The meaning of EG, EU, AF can be expressed using least and greatest fixpoints (lfp, gfp) of some F (monotone)

On the other hand, procedures for these connectives encode fixpoint computations

⇒ can prove correctness of our procedures using fixpoint theory.

Notation: $F^i(X) \stackrel{def}{=} \underbrace{F(F(F\ldots(F}_{i \text{ times}}(X))))$

Consider sequence:

$$(F^1, F^2, F^3, \ldots)$$

For <u>monotone</u> F, $\exists n_0$ such that $F_{n_0} = F_{n_0+1}$

i.e. the sequence always stabilizes

Not the case for functions which are not monotone, e.g. G

Th (*) Let S be a set $\{s_0, \dots, s_n\}$ with $n+1$ elements.

If $F : \mathcal{P}ow(S) \to \mathcal{P}ow(S)$ is monotone then

- $F^{n+1}(\emptyset)$ is the (unique) lfp of F
- $F^{n+1}(S)$ is the (unique) gfp of F

---

not the Knaster-Tarski theorem, not even a particular case of it.

Knaster-Tarski theorem:

A monotone operator on a complete lattice has a complete lattice of fixpoints; thus, the least and the greatest fixpoint.

Obtained by Tarski in 1936 (or 1939), published in a joint paper with Knaster in 1955.

Proof of Th (*)

$\emptyset \subseteq F(\emptyset)$, obviously.

By monotonicity of F,

$$F(\emptyset) \subseteq F(F(\emptyset)) = F^2(\emptyset)$$

again, $F^2(\emptyset) \subseteq F^3(\emptyset)$

etc.

By induction, conclude $\forall i > 0$

$$F^1(\emptyset) \subseteq F^2(\emptyset) \subseteq \ldots \subseteq F^i(\emptyset)$$

__Claim__: One of these is a fixpoint of F, call it $F^j(\emptyset)$, $1 \leq j \leq n+1$,

i.e., $F(\underline{F^j(\emptyset)}) = \underline{F^j(\emptyset)}$

assume, towards a contr., that none of these is a fixpoint of F

Then $F^1(\emptyset)$ must contain at least one element

$$(\emptyset \neq F(\emptyset) \text{ by assump.});$$

$F^2(\emptyset)$ must contain at least one more element than $F^1(\emptyset)$

($F^1(\emptyset)$ is not a fixpoint by assump., i.e.,

$$\underline{F^1(\emptyset)} \neq \underbrace{F(\underline{F^1(\emptyset)})}_{F^2(\emptyset)}$$

....

$F^{n+2}(\emptyset)$ must contain at least $n+2$ elements

But this is impossible since we only have $n+1$ elements.

$\Rightarrow F^j(\emptyset)$ is a fixpoint of F for some j, $1 \leq j \leq n+1$.

$\Rightarrow$ In particular $F^{n+1}(\emptyset)$ is a fixpoint of F

Let's show it is the least one.

Suppose X is another fixpoint of F.

We'll show $F^{n+1}(\emptyset) \subseteq X$,

~~i.e., X is not the least~~ one.

$\emptyset \subseteq X$, obviously.

$F(\emptyset) \subseteq F(X) = X$

by monot. of F

since X is a fixpoint

So, $F(\emptyset) \subseteq X$

etc. $\boxed{F^i(\emptyset) \subseteq X \quad \forall i > 0}$

by induction.

In particular,

$$F^{n+1}(\emptyset) \subseteq X.$$

So, for any fixpoint X,

$F^{n+1}(\emptyset)$ is included in it,

thus, it is the <u>least one</u>.

The proof for the gfp is similar.