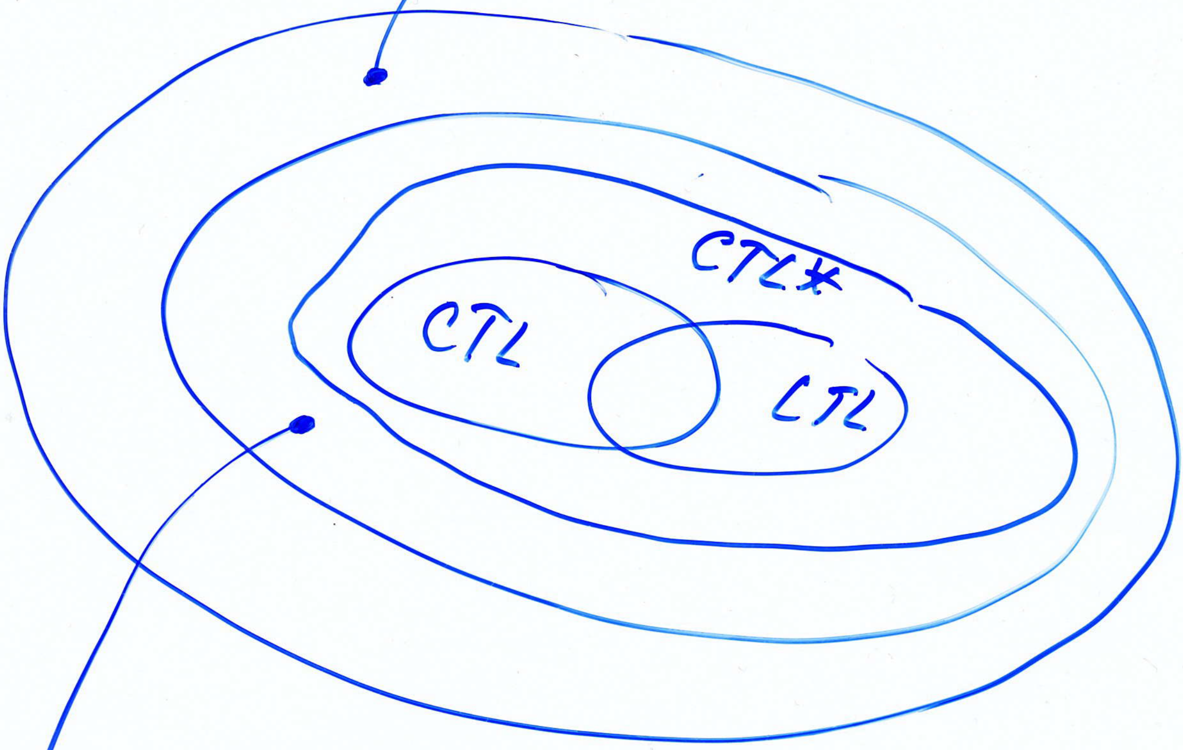
$L_\mu$ the $\mu$-calculus




$L_{\mu_1}$

the alternation-free
$\mu$-calculus

$L_\mu$: have expressions for lfp ($=\mu$)
and gfp ($=\nu$) in <u>the language</u>

# A Relational mu-calculus

Recall: we need lfp and gfp operators to compute

$[\![ EF\varphi ]\!]$, $[\![ EG\varphi ]\!]$ etc.

$\Rightarrow$ introduce a general calculus to deal with fixpoints

bool vars

<u>Def</u>: $v ::= x \mid Z$

$$f ::= 0 \mid 1 \mid v \mid \bar{f} \mid f_1 + f_2 \mid f_1 \cdot f_2 \mid f_1 \oplus f_2$$

$$\exists x.\, f \mid \forall x.\, f \mid \underset{\text{mu}}{\mu} Z.\, f \mid \underset{\text{nu}}{\nu} Z.\, f \mid f[\hat{x} := \hat{x}$$

We require that $Z$ in $f$ occurs positively (i.e., within an even # of $-$) to ensure monotonicity. (a sufficient cond.)

## Convention on binding priorities:

"$-$" and $[\hat{x} := \hat{x}']$ have the highest priority

then $\exists x$, $\forall x$

then $\mu Z$, $\nu Z$

then $\bullet$

then $+$ and $\oplus$

Def. a valuation $\rho$ for $f$ is an assignment of 0 or 1 to all variables of $f$

Notation:

$\rho(v)$ – value of $v$ under $\rho$

$$\left.\begin{array}{l}\rho[v \mapsto 0] \\ \\ \rho[v \mapsto 1]\end{array}\right\}$$ "updated" valuatio[n] act just like $\rho$, but the value of $v$ is "updated" (to 0 or 1 respect.)

short notation for valuation $\rho$

$$\boxed{(x, y, z) \Rightarrow (1, 0, 1)}$$

means $\rho(x) = 1$

$\rho(y) = 0$

$\rho(z) = 1$

and $\rho(v) = 0$ for all other variables

We can update $\rho$, for example:

$\rho[x \mapsto 0]$ is $(x, y, z) \Rightarrow (0, 0, 1)$

## Def. ($\rho \models f$) without the fixpoint subformulas for now

- $\rho \not\models 0$
- $\rho \models 1$
- $\rho \models v$ iff $\rho(v) = 1$
- $\rho \models \overline{f}$ iff $\rho \not\models f$
- $\rho \models f \cdot g$ iff $\rho \models f$ and $\rho \models g$
- $\rho \models f + g$ iff $\rho \models f$ or $\rho \models g$
- $\rho \models f \oplus g$ iff $\rho \models (f \cdot \overline{g} + \overline{f} \cdot g)$
- $\rho \models \exists x . f$ iff

  $\rho[x \mapsto 0] \models f$ <u>or</u> $\rho[x \mapsto 1] \models f$
- $\rho \models \forall x . f$ iff

  $\rho[x \mapsto 0] \models f$ <u>and</u> $\rho[x \mapsto 1] \models f$

- $\rho \models f[\hat{x} := \hat{x}']$ iff $\underbrace{\rho[\hat{x} := \hat{x}']} \models f$

same as $\rho$,
but for each $x_i$,
it assigns $\rho(x_i')$

(i.e. $\rho[x_i \to \rho(x_i')]$
for each $x_i$ in $\hat{x}$ )

$$\rho : \quad \begin{array}{l} x_1 \longrightarrow 0 \\ x_2 \longrightarrow 1 \\ x_1' \longrightarrow 1 \\ x_2' \longrightarrow 1 \end{array}$$

$$\rho[\hat{x} := \hat{x}'] : \quad \begin{array}{l} x_1 \longrightarrow 1 \\ x_2 \longrightarrow 1 \\ x_1' \longrightarrow 1 \\ x_2' \longrightarrow 1 \end{array}$$

Example: Let $\rho(x_1') = 0$
$\rho(x_2') = 1$

Evaluate

$$\rho \overset{?}{\models} \overbrace{(x_1 + \overline{x_2})}^{f}[\hat{x} := \hat{x}']$$

$$\Updownarrow$$

$$\underline{\rho[\hat{x} := \hat{x}']} \models \overset{\overbrace{0 + \overline{1}}^{0}}{(x_1 + \overline{x_2})}$$

$$\shortparallel$$

$$\rho[x_1 \to \underbrace{\rho(x_1')}_{0},\ x_2 \to \underbrace{\rho(x_2')}_{1}] \quad (*)$$

**Answer: no**

We could, instead of the substitution *

continue on the structure of the formula (function)

$$\Updownarrow$$

$\rho[\hat{X} := \hat{X'}] \models X_1$ or

$\rho[\hat{X} := \hat{X'}] \models \overline{X}_2$

$$\Updownarrow$$

$\rho[x_1 \to \underbrace{\rho(x_1')}_{0}] \models \overset{0}{\widehat{X_1}}$ or

$\rho[x_2 \to \rho(x_2')] \models \underbrace{\overline{X}_2}_{0}$

false.

Extend $\models$ to fixpoint operators.

Construct sequence:

$\mu_0 Z.f,\ \mu_1 Z.f, \ldots$ $\mu_i$ similar to lfp

such that

$$\mu_0 Z.f \stackrel{def}{=} 0$$

$$\mu_{m+1} Z.f \stackrel{def}{=} f[\mu_m Z.f / Z] \quad \text{for } m > 0$$

Substitute the result of the previous step for $Z$ in $f$ (*)

" " " in f

use "free" occurrences of $Z$ in $f$
(= not "bound" by $\nu, \mu$) another

Define :

$\rho \models \mu Z.f$ iff

$\rho \models \mu_m Z.f$ for some $m$

Strategy: find the smallest such $m$.

Examples of substitution (*)

$$1)\ \left(x_1 + \exists x_2 . (z \cdot x_2)\right)[\bar{x}_1 / z]$$

$$\parallel$$

$$x_1 + \exists x_2 . (\bar{x}_1 \cdot x_2)$$

$$2)\ \mu z . \overbrace{(\mu z . x_1 + z) \cdot (x_1 + \exists x_2 . (z \cdot x_2))}^{\text{free in } f}$$

$$[\bar{x}_1 / z]$$

bounded by $\mu$

$$\parallel$$

$$(\mu z . x_1 + z) \cdot (x_1 + \exists x_2 . (\bar{x}_1 \cdot x_2))$$

Example: ~~Show~~ Determine: $\rho \overset{?}{\models} \mu Z. Z$

$\rho$ is arbitrary

0) try $\rho \models \underbrace{\mu_0 Z. Z}_{0 \text{ by def.}}$

so $\rho \not\models \mu_0 Z. Z$

1) try $\rho \models \mu_1 Z. Z$

$$\| $$

Our function $Z[\mu_0 Z. Z / Z]$

$$\| $$

$$\mu_0 Z. Z$$

$$\| $$

$$0$$

so $\rho \not\models \mu_1 Z. Z$

And so on.

In fact, $\mu_m Z.\, Z = \mu_0 Z.Z$

for all $m \geq 0$

So $\rho \not\models \mu Z.Z$

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Define $\underline{\rho \models \nu Z.f}$:

Define sequence

$\nu_0 Z.f,\ \nu_1 Z.f,\ \nu_2 Z.f, \ldots$ by

$$\nu_0 Z.f \stackrel{\text{def}}{=} 1$$

$$\nu_{m+1} Z.f \stackrel{\text{def}}{=} f[\nu_m Z.f / Z],\ m \geq 0$$

Similar to $\mu$, but start from 1

Define:

$\rho \models \nu z.f$ iff $\rho \models \nu_m z.f$

for <u>all</u> $m \geq 0$

Analogy: $EG\varphi$

requires that $\varphi$ is true on <u>all</u> elements of the path.

Intuitively,

$\rho \not\models \mu Z.f$ until and if

$\rho \models \mu Z.f$ is proven

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Note: We do not discuss more complex case of alternation of $\mu$ and $\nu$ as in e.g.

$$\nu Z.\, f(Z, \mu X.\, g(X, Z))$$

(mutual dependencies between $\nu$ and $\mu$)