Logical consequence

(= logical (tautological) implication)

Logical Equivalence

A very important notion.

Def. Let $\Gamma = \{A_1, A_2, \ldots\}$ be a set of formulas, B a formula, S a set of atoms containing every atom which appears in Γ or B.

B is a logical consequence of Γ (or, Γ logically entails B), $\Gamma \models B$, if every truth assignment for S which satisfies Γ also satisfies B.

$$\{\neg Q,\ P \vee Q\} \overset{?}{\models} P$$

P	Q	$\neg Q$	$P \vee Q$
0	0	1	0
0	1	0	1
1	0	1	1
1	1	0	1

Notation: $\models A$

means that A is valid (true under every t.a.)

$$\{P \wedge Q\} \overset{?}{\models} Q$$

we usually omit $\{\ \}$ if it is just one f-la

yes

P	Q	
0	0	
0	1	
1	0	
1	1	

$$P \vee Q \overset{?}{\models} Q$$

no

Proposition Let Γ be a set of f-las, B a formula.

$$\boxed{\Gamma \models B \text{ iff } \Gamma \cup \{\neg B\} \text{ is unsatisfiable}}$$

($\Rightarrow$)

Proof: Assume $\Gamma \models B$. Consider an arbitrary t.a. τ (tau) (for all atoms in Γ, B)

Two cases. 1) $\tau \not\models \Gamma$, then $\tau \not\models \Gamma \cup \{\neg B\}$.

2) $\tau \models \Gamma$, By assumption ($\Gamma \models B$), and def. of logical consequence $\tau \models B$.

Thus, $\tau \not\models \neg B$ (by def. of $\tau \models \dots$)

So $\tau \not\models \Gamma \cup \{\neg B\}$

Since τ was arbitrary, $\Gamma \cup \{\neg B\}$ is unsat. (there is no t.a. that satisfies it).

($\Leftarrow$) is similar. See the notes.

Def. Formulas A and B are logically equivalent iff $A \models B$ and $B \models A$. Notation: $A \equiv B$

$$(A \wedge B) \equiv (B \wedge A)$$

$$\neg\neg A \equiv A$$

$$((A \vee B) \vee C) \equiv (A \vee (B \vee C))$$

$$(A \wedge A) \equiv A$$

$$\neg (A \wedge B) \equiv \neg A \vee \neg B$$

etc.

The textbook introduces connectives : $\neg, \wedge, \vee, \rightarrow$ $(\leftrightarrow, \oplus)$,

but not all of them are necessary.

Some connectives are expressible using the other ones:

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \oplus q \equiv \neg (p \leftrightarrow q)$$

exclusive or

$$\equiv (p \vee q) \wedge \neg (p \wedge q)$$

e.g. $\{\neg, \vee, \wedge\}$, $\{\neg, \wedge\}$, ...

Adequate set of connectives: contains connectives such that all other connectives are expressible using them

So far:

- semantic entailment
  (= logical, tautological entailment,
  = logical implication)

$\varphi_1 \ldots \varphi_n \models \psi$ not a connective!

a meta-logic notation

do not confuse with $\rightarrow$, material implication, a connective.

- semantic equivalence

$$\varphi_1 \equiv \varphi_2$$

do not confuse with $\leftrightarrow$

- complete set of connectives